Stability in Random Coalition Formation

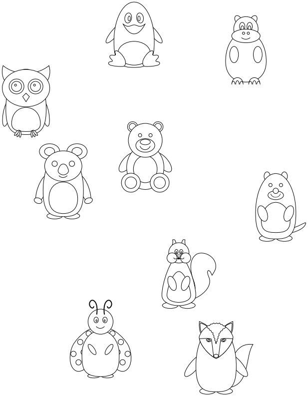
Games, Mechanisms and Social Networks seminar, Warsaw

Sonja Kraiczy

Joint work with Martin Bullinger

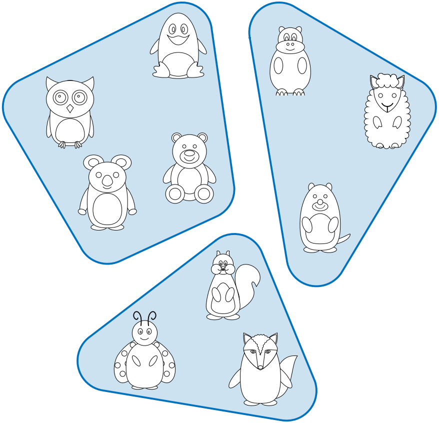
UNIVERSITY OF OXFORD

Coalition Formation



Drèze and Greenberg (ECMA 1980)

Coalition Formation



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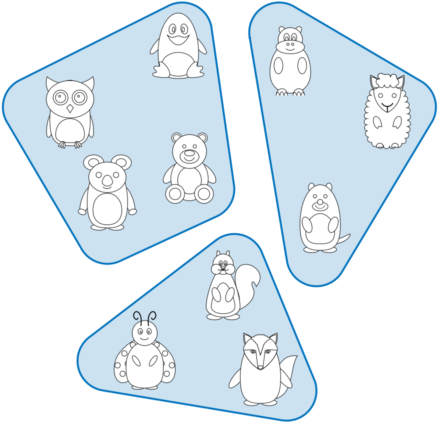


How to obtain desirable coalition structures algorithmically?

Drèze and Greenberg (ECMA 1980)

Applications

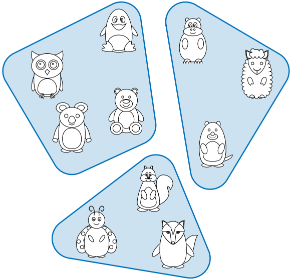
Team allocation
Cooperative game theory

Clustering
Machine learning

Community detection
Social sciences

Hedonic Games: Formal Model

- Set N of n agents
- Agent $i \in N$ expresses preference order over coalitions
- Output: coalition structure (= partition) of agents
- Representation issues: 2^{n-1} possible coalitions

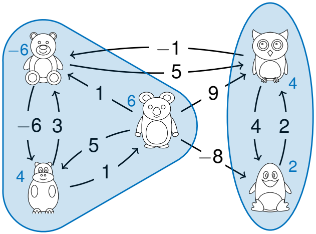


Drèze and Greenberg (ECMA 1980)

Additively Separable Hedonic Games

- Preferences encoded by utility functions $u_i \colon N \to \mathbb{Q}$
- Induces hedonic game where

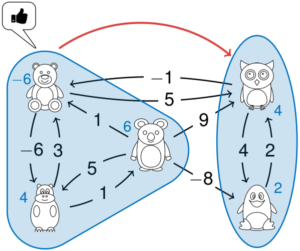
$$C \succsim_i C' \iff \sum_{j \in C} u_i(j) \geq \sum_{j \in C'} u_i(j)$$



Bogomolnaia and Jackson (GEB, 2002)

Single-Deviation Stability

Stable partition $\hat{=}$ no beneficial deviation by single agent



- Nash deviation: beneficial deviation to other coalition
- Nash-stable: there are no Nash deviations

Drèze and Greenberg (ECMA 1980), Dimitrov and Sung (JME 2007)

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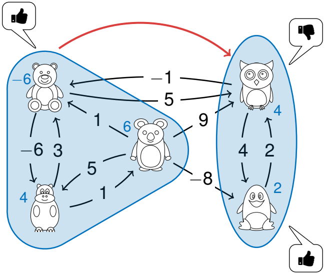

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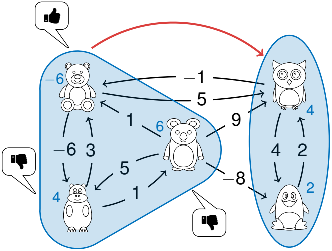
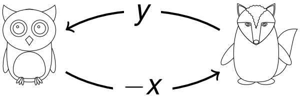


- Nash deviation: beneficial deviation to other coalition
- Nash-stable: there are no Nash deviations
- Individually stable: every Nash deviation blocked by agent in joined coalition
- Contractually Nash-stable: every Nash deviation blocked by agent in abandoned coalition

Drèze and Greenberg (ECMA 1980), Dimitrov and Sung (JME 2007)

The Run and Chase Instance

- Consider a hedonic game where $N = \{\text{Owl}, \text{Fox}\}$
- Owl prefers to be alone over the grand coalition
- Fox prefers the grand coalition over being alone
- Nash-stability too demanding? Unreasonable?



Complexity of Stability

Theorem

It is NP-complete to decide if there exists a

- *Nash-stable partition (Sung and Dimitrov, EJOR 2010),*
- *individually stable partition (Sung and Dimitrov, EJOR 2010),*
- *contractually Nash-stable partition (Bullinger, MFCS 2022)*

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- Hardness for Nash-stability even if utilities restricted to $\{-x, y\}$ for $x \geq y \geq 0$ (Brandt et al., AAAI 2022)
- Reduced instances seem to be artificial corner cases

Existence of Stable Outcomes

- Nash-stable partitions exist for symmetric utilities (Bogomolnaia and Jackson, GEB 2002)
  - PLS-complete to compute (Gairing and Savani, MOR 2019)

Existence of Stable Outcomes

- Nash-stable partitions exist for symmetric utilities (Bogomolnaia and Jackson, GEB 2002)
  - PLS-complete to compute (Gairing and Savani, MOR 2019)
- Individually stable and contractually Nash-stable partitions exist for $\{-x, y\}$-utilities (Brandt et al., AAAI 2022)
  - Natural dynamics runs in polynomial time

Stability in Random Games

Question: Do stable coalition structures *typically* exist for many agents?

- Random hedonic game $H(n, \mathcal{D})$
  - Set of n agents
  - Pairwise utility sampled i.i.d. from a distribution $\mathcal{D}$
- Investigate probability of property $\mathfrak{P}$ (e.g., stability) when n tends to infinity:

$$\lim_{n \to \infty} \mathbb{P}\left(\mathcal{H}(n, \mathcal{D}) \text{ satisfies } \mathfrak{P}\right) = ?$$

First Observations

Grand coalition (all agents in one large coalition)

- Nash-stable if $\mathcal{D}$ has positive mean, e.g., $\mathcal{D} = U(-1, 2)$,
- contractually Nash-stable if positive weight on positive utility,
- not individually stable if $\mathcal{D}$ has mean 0, e.g., $\mathcal{D} = U(-1, 1)$

Main Theorem

Theorem (Bullinger and Kraiczy, 2024)

Let $\mathcal{D} = U(-1, 1)$.
Then, $\lim_{n\to\infty} \mathbb{P}\left(H(n, \mathcal{D}) \text{ admits Nash-stable partition}\right) = 0$.
Moreover, there exists an efficient algorithm $\mathcal{A}$ *such that*

- $\lim_{n\to\infty} \mathbb{P}\left(\mathcal{A}(H(n, \mathcal{D})) \text{ individually stable}\right) = 1$, *and*
- $\lim_{n\to\infty} \mathbb{P}\left(\mathcal{A}(H(n, \mathcal{D})) \text{ contractually Nash-stable}\right) = 1$.

$H(n, \mathcal{D})$: random hedonic game

Exit Denial and Entry Denial

- Consent prevents deviations from / to large coalitions
  - Every agent denied to leave / join
- Individual stability requires individual rationality (nonnegative utilities)

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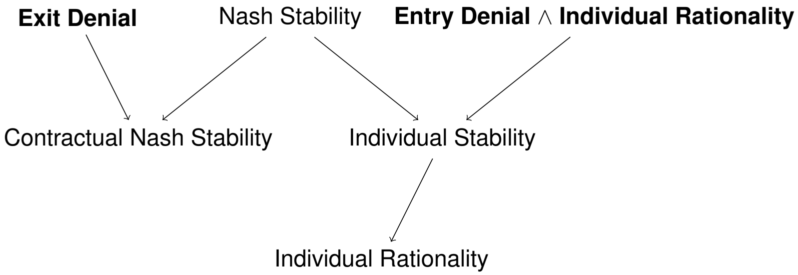


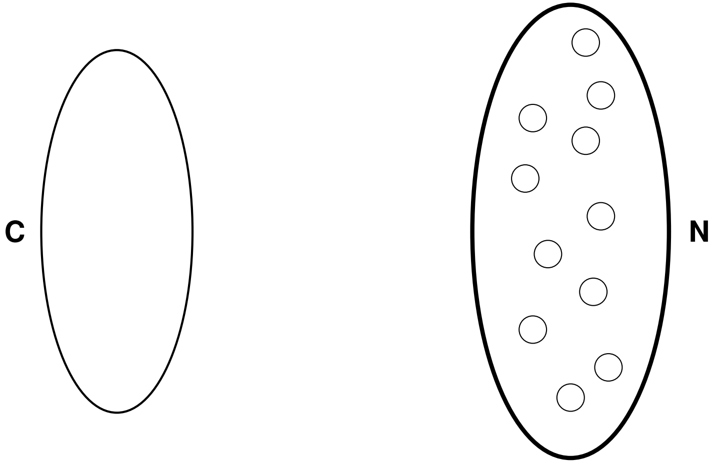
Goal of Algorithm

Construct partition that satisfies

- Individual Rationality
- Entry-Denial
- Exit-Denial

Stage 1: Greedy Clique Formation

Form coalitions with high mutual utility



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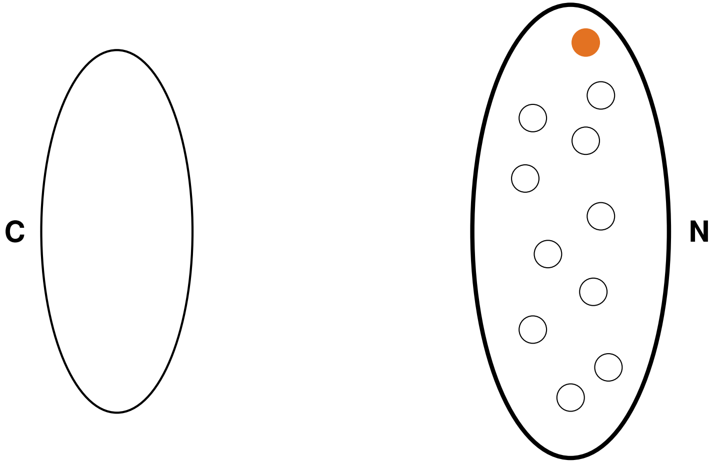

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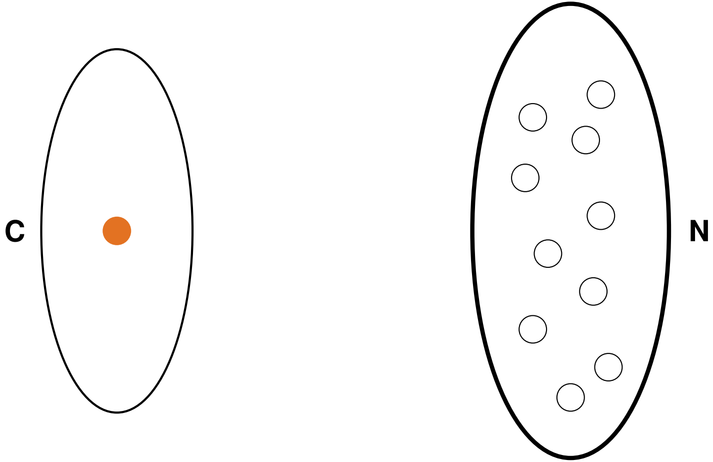

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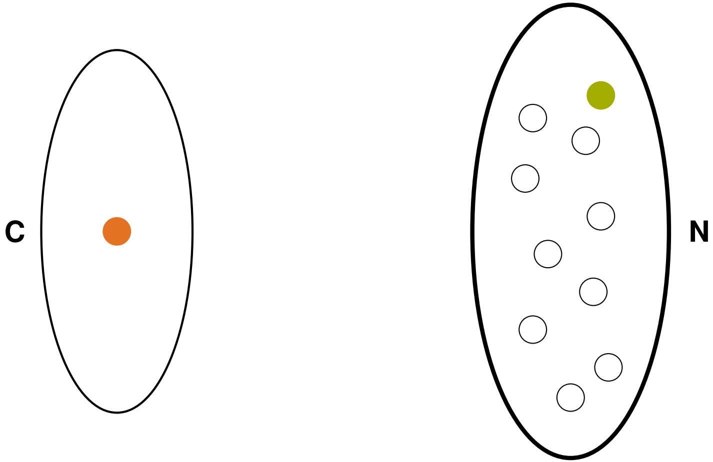

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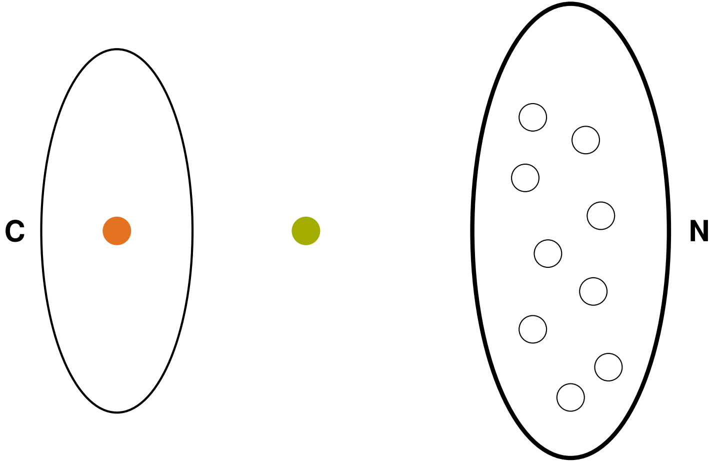

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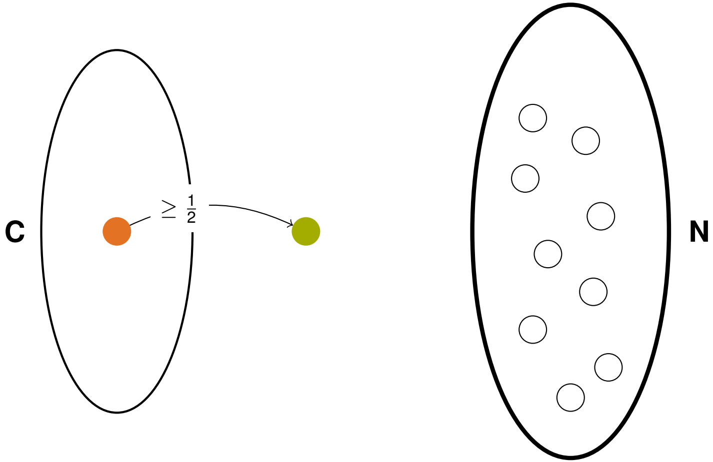

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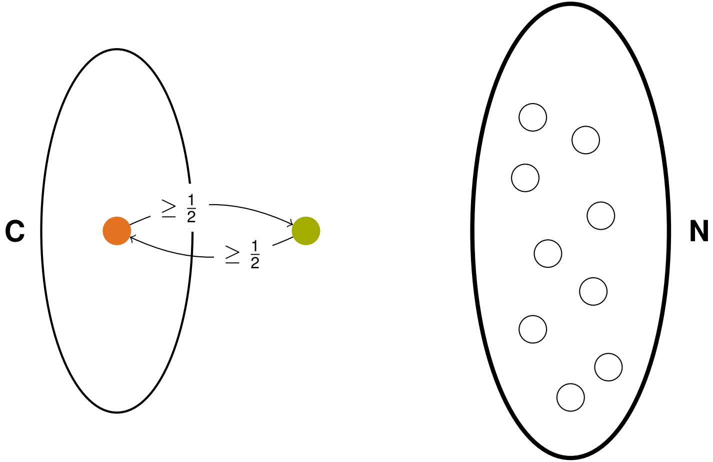

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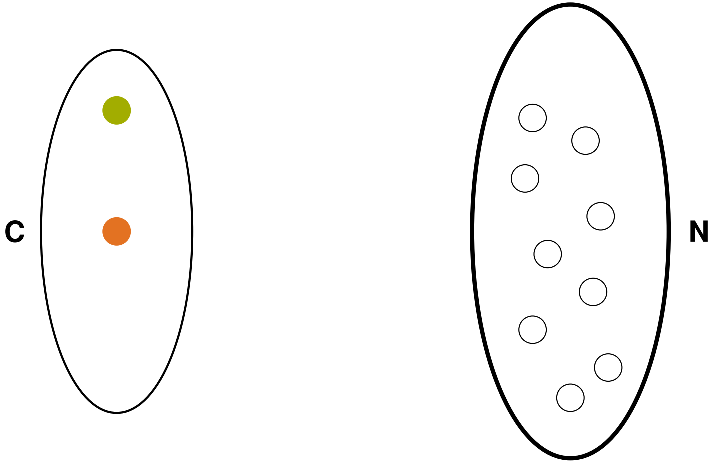

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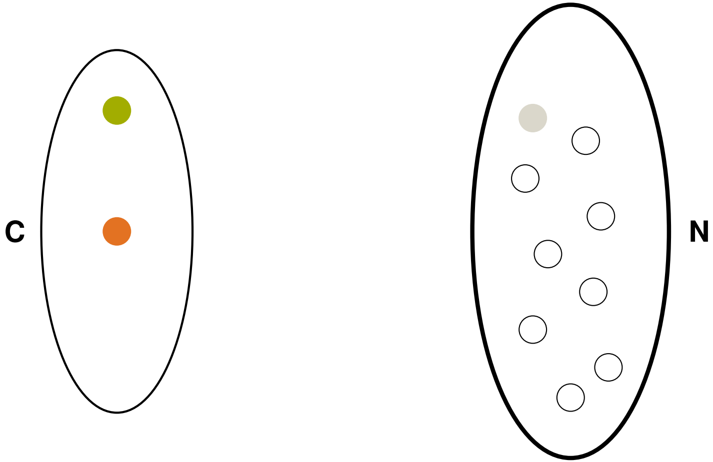

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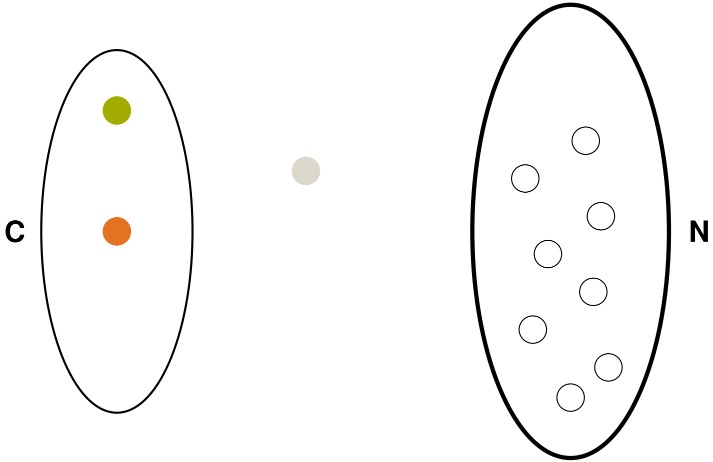

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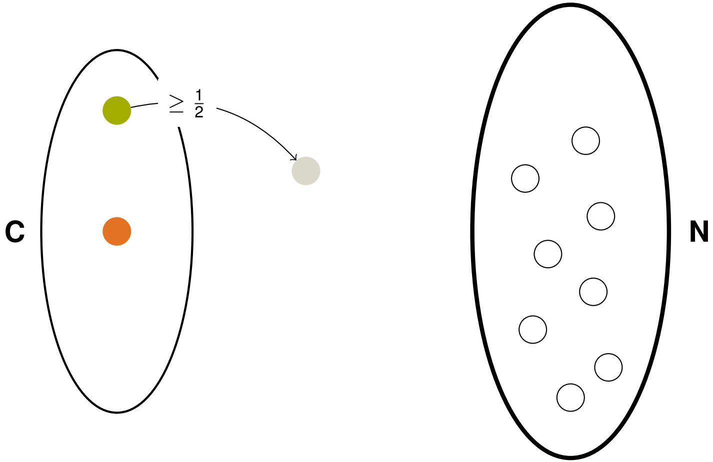

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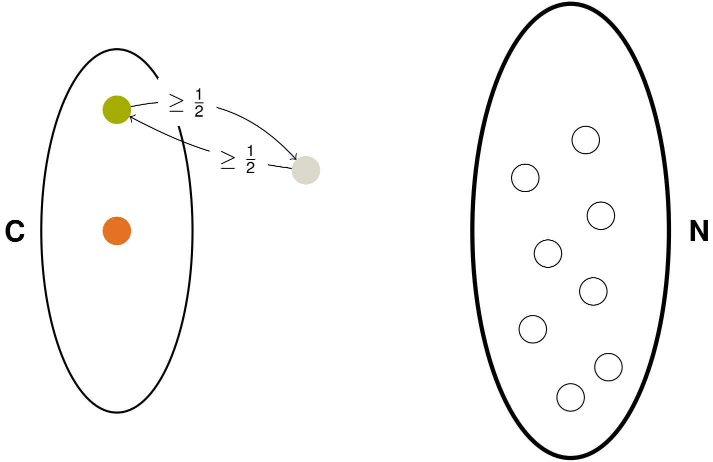

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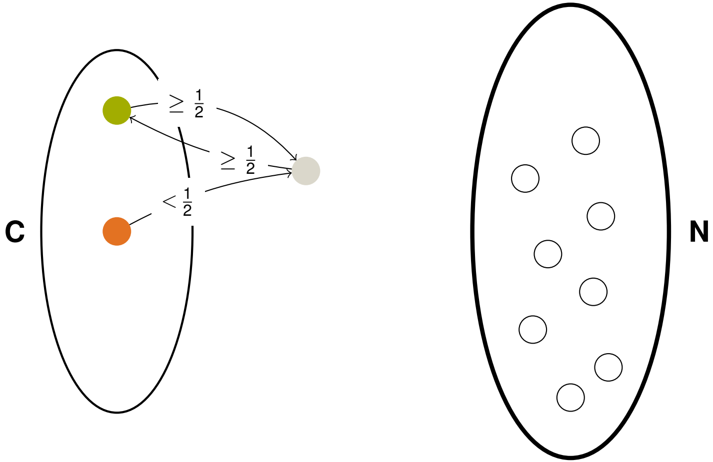

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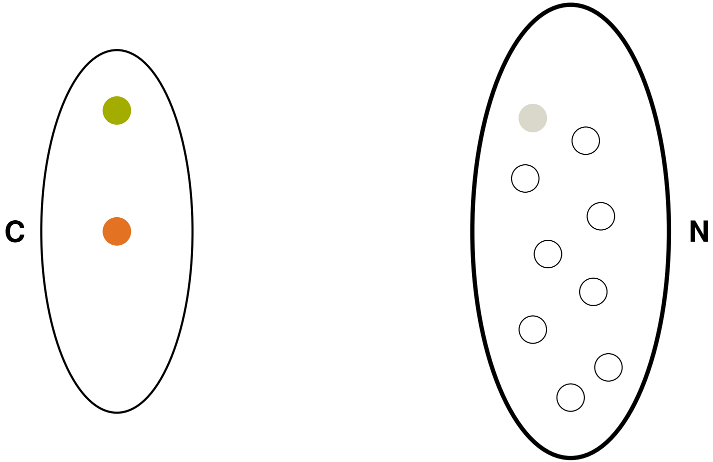

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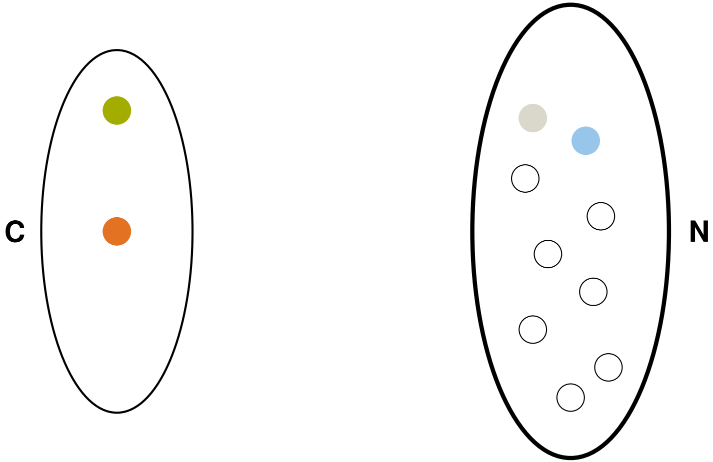

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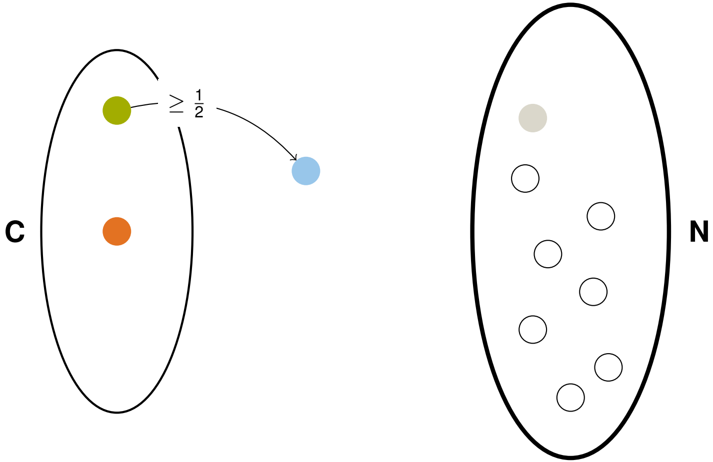

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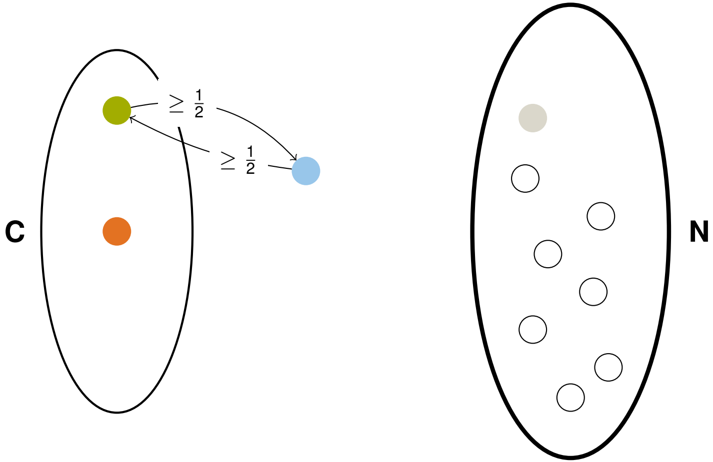

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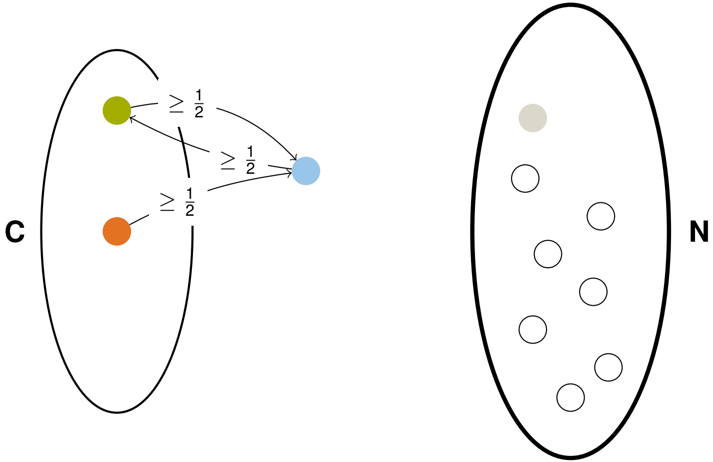

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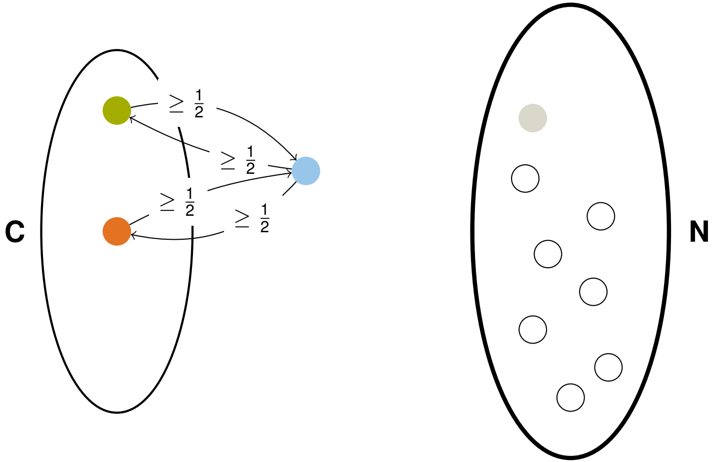

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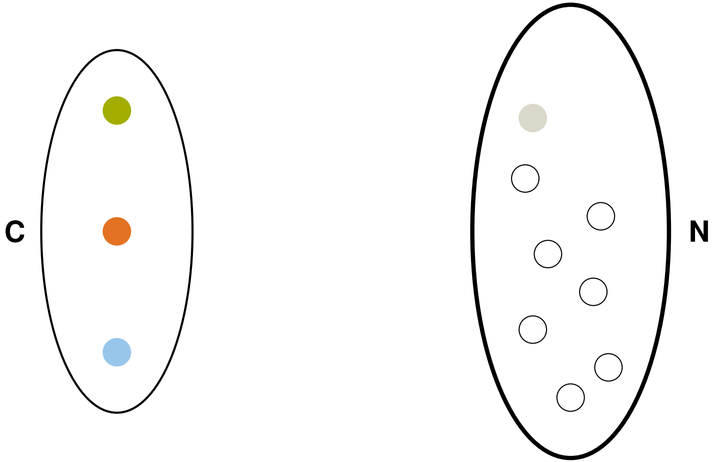


Performance of Stage 1

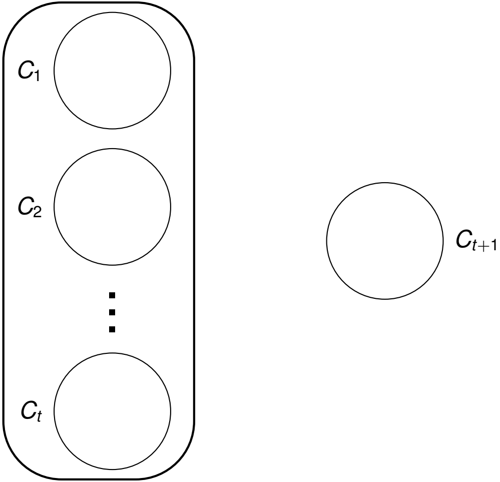
Theorem

With high probability, all except at most $\frac{n}{\log_{16}^2 n}$ agents are assigned to coalitions of size $\left\lceil \frac{\log_{16} n}{2} \right\rceil$.

- Good for individual rationality
- (Nonsingleton) coalitions fail entry denial
- Idea: enlarge coalitions while losing little utility

Stage 2: Greedy Clustering

Merge coalitions with small utility loss



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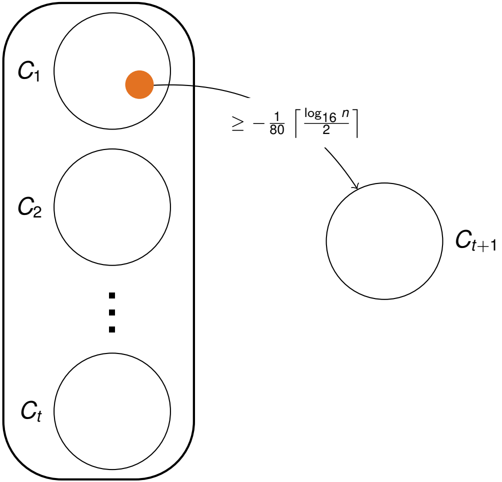

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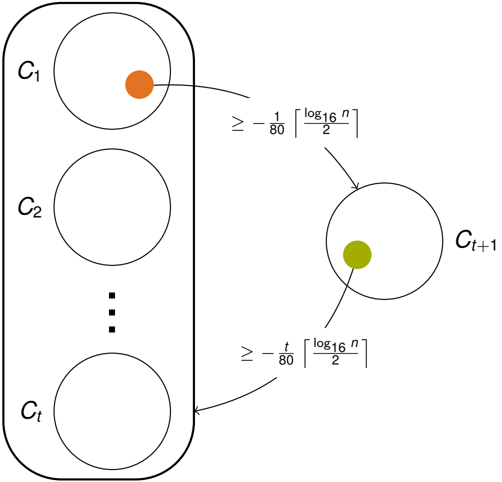


Performance of Stage 2

Theorem

With high probability, all except at most $20\frac{n}{\log_{16}^2 n} + \alpha \left\lceil \frac{\log_{16} n}{2} \right\rceil$ *agents are assigned to coalitions of size* $20 \left\lceil \frac{\log_{16} n}{2} \right\rceil$*.*

- Split agents set into 20 subsets
- Run Stage 1 for each individually
- Merge 20 coalitions each
- Only a finite number of Stage 1 coalitions not merged

Stage 3: Assigning Remainder Agents

Theorem

With high probability, the remainder agents can be added to coalitions for which

- *they receive positive utility,*
- *no utility values revealed in Stage 2.*

- First property: individual rationality
- Second property: exit denial

Main Theorem 1

Theorem (Bullinger and Kraiczy, 2024)

Let $\mathcal{D} = U(-1, 1)$. Then, there exists an efficient algorithm $\mathcal{A}$ such that

- $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}(H(n, \mathcal{D}))$ *individually stable*$) = 1$, *and*
- $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}(H(n, \mathcal{D}))$ *contractually Nash-stable*$) = 1$.

- Individual rationality: Stages 1 and 3
- Entry denial: Stage 2
- Exit denial: Stages 2 and 3

$H(n, \mathcal{D})$: random hedonic game

Main Theorem 2

Theorem (Bullinger and Kraiczy, 2024)

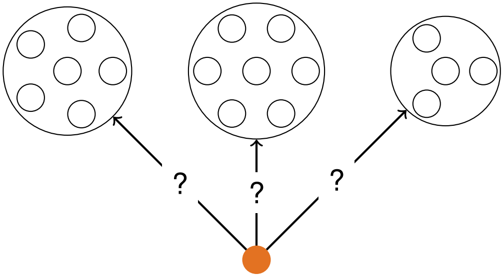
Let $\mathcal{D} = U(-1, 1)$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}\left(H(n, \mathcal{D}) \text{ admits Nash-stable partition}\right) = 0.$$

- Sophisticated counting argument
- Bound probability of Nash stability given a fixed number of coalitions
- Multiply with Stirling number of second kind

$H(n, \mathcal{D})$: random hedonic game

Proof Idea



- Nash stability captured by comparing sums of i.i.d. random variables

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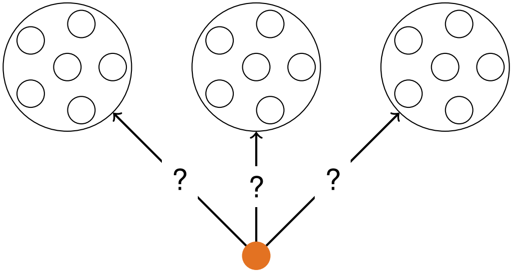

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- Probability bounded by case of identical-size coalitions

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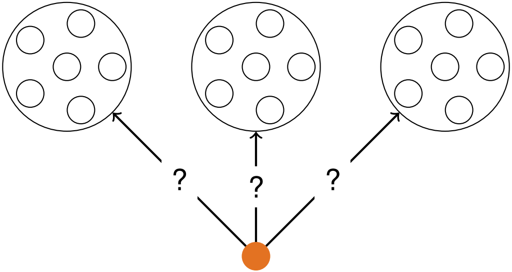

- Nash stability captured by comparing sums of i.i.d. random variables
- Probability bounded by case of identical-size coalitions
- Challenge: agents are themselves part of a coalition

Conclusion

- Random model of coalition formation
- High probability analysis in large agent limit
- Nash stability fails to exist
- Individual stability and contractual Nash stability derived by efficient algorithm

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Future directions

- Other probability distributions
- Other (stability) concepts
- Other game classes