A Generalised Theory of Proportionality in Collective Decision Making

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What we will talk about

- The Model
- Proportionality
- PAV
- Priceable Outcomes
- Extensions of the model

Model and Notation

- $C = \{c_1, \ldots, c_m\}$ - set of m candidates
- $N = \{1, 2, \ldots, n\}$ - set of n voters
- $\mathcal{F} \subseteq 2^C$ - family of feasibility sets
- $A = (A_1, \ldots, A_n)$ - collection of approval ballots
- $u_i(W)$ - utility of voter i from winning set W

Generalized models

Committee elections: $\mathcal{F} = \{W \subseteq C : |W| = k\}$

Public Decisions: $\mathcal{F} = \{W \subseteq C : |W \cap C_r| = 1\}$

Committee elections with disjoint attributes:

$\mathcal{F} = \{W \subseteq C : |W| = k \text{ and } q_r^{\perp} \leq |W \cap C_r| \leq q_r^{\top} \text{ for each r} \in [z]\}$

Matroid constraints

Exchange property:

For each $X, Y \in \mathcal{F}$ such that $|X| < |Y|$,
there exists $c \in Y \setminus X$ such that $X \cup \{c\} \in \mathcal{F}$

Proportionality

- Base Extended Justified Representation (BEJR)
- Extended Justified Representation (EJR)
- Restrained EJR

Base Extended Justified Representation

A group of voters S deserves l if for each feasible set T either:

- Exists X $\subseteq \cap A_i$, $|X| \geq l$ such that $T \cup X \in \mathcal{F}$
- $\frac{|S|}{n} > \frac{l}{|T| + l} \iff \frac{|S|}{n - |S|} > \frac{l}{|T|}$

An outcome W satisfies BEJR if for each l and for each group of voters S that deserves l candidates there exists a voter $i \in S$ for which $u_i(W) \geq l$

Example - Public Decisions

S - 30% of voters

p issues, S deserves $\lfloor 0.3 \cdot p \rfloor$

If $|T| \leq p - \lfloor 0.3 \cdot p \rfloor$:

We can add p issues to T

so that $T \cup X \in \mathcal{F}$

If $|\mathrm{T}| >$ p - $\lfloor 0.3 \cdot p \rfloor$:

$$\frac{\lfloor 0.3 \cdot p \rfloor}{|T| + \lfloor 0.3 \cdot p \rfloor} < \frac{0.3 \cdot p}{p} \leq \frac{|S|}{n}$$

Base Extended Justified Representation

- For each election there exists an outcome satisfying BEJR
- Average utility is quite high
- Generalisation of strong proportionality axioms in each of the generalised models
- Never contradicts Pareto optimality

Extended Justified Representation

A group of voters S deserves l if for each set $T \subseteq W$ either:

- Exists X $\subseteq \cap A_i$, $|X| \geq l$ such that $T \cup X \in \mathcal{F}$
- $\frac{|S|}{n} > \frac{l}{|T| + l} \iff \frac{|S|}{n - |S|} > \frac{l}{|T|}$

An outcome W satisfies EJR if for each l and for each group of voters S that deserves l candidates there exists a voter $i \in S$ for which $u_i(W) \geq l$

Extended Justified Representation

- Is stronger than both BEJR and Restrained EJR
- Open problem whether a satisfying output always exists
- Can contradict Pareto optimality for non-matroid constraints
- Also guarantees high average satisfaction

Restrained EJR

- Made independently of BEJR and EJR, authors mostly prefer those
- Can be always satisfied
- Exclusive with Pareto Optimality outside matroid constraints

PAV

$$score_{PAV}(W) = \sum_{i \in \mathbb{N}} H(|W \cap A_i|), \quad \text{where } H(k) = \sum_{j=1}^{k} \frac{1}{j}$$

- Chooses outcome with the highest score
- NP-hard to compute
- Excellent proportionality properties in other models
- With matroid constraints, satisfies EJR
- With non-matroid constraints, fails BEJR

Phragmén’s Sequential Method

- Each candidate costs 1 dollar
- Candidate is bought when all their supporters have the dollar
- We start with empty outcome and 0 dollars
- Voters gain money linearly and spend it when possible
- After each purchase we eliminate candidates that would make the outcome not feasible

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- Candidate is bought when all their supporters have the dollar
- We start with empty outcome and 0 dollars
- Voters gain money linearly and spend it when possible
- After each purchase we eliminate candidates that would make the outcome not feasible

- Can be computed in polynomial time
- Fails EJR in committee elections, so BEJR in our model

Base Proportional Justified Representation

A group of voters S deserves l if for each feasible set T either:

- Exists X $\subseteq \cap A_i$, $|X| \geq l$ such that $T \cup X \in \mathcal{F}$
- $\frac{|S|}{n} > \frac{l}{|T| + l} \iff \frac{|S|}{n - |S|} > \frac{l}{|T|}$

An outcome W satisfies BPJR if for each l and for each group of voters S that deserves l candidates there are at least l candidates from $\bigcup_{i \in S} A_i$ in W

Proportional Justified Representation

A group of voters S deserves l if for each set $T \subseteq W$ either:

- Exists X $\subseteq \cap A_i$, $|X| \geq l$ such that $T \cup X \in \mathcal{F}$
- $\frac{|S|}{n} > \frac{l}{|T| + l} \iff \frac{|S|}{n - |S|} > \frac{l}{|T|}$

An outcome W satisfies PJR if for each l and for each group of voters S that deserves l candidates there are at least l candidates from $\bigcup_{i \in S} A_i$ in W

Phragmén’s Sequential Method

- With matroid constraints, satisfies PJR
- With matroid constraints, same average utility guarantee as with BPJR
- With non-matroid constraints, fails BPJR, but still offers good approximation:

$$\mathrm{PJR} \cdot \left(1 - \frac{|S|}{n}\right)$$

Stable Priceable Outcomes

We find candidate prices $\{\pi_c\}_{c \in C}$ and payment functions $\{p_i\}_{i \in \mathbb{N}}$ such that:

- The voters pay only for the selected candidates
- The total payment for each candidate must be equal its price
- The outcome maximises the total price
- For each unselected candidate c:

$$\sum_{i \in N(c)} \max\left(r_i,\ \max_{c' \in W} p_i(c')\right) \leq \pi_c \quad \text{where } r_i = 1 - \sum_{c' \in W} p_i(c')$$

Stable Priceable Outcomes

- Solutions not always exist
- With matroid constraints all outcomes satisfy EJR
- With non-matroid constraints outcomes satisfy EJR if all candidate prices are equal
- Can be computed using linear programming

General Monotone Utility Functions

- An extension where the utilities for candidates are different for different voter
- Each voter assigns utility to each outcome, must be monotone
- Generalisation of BPJR and others exists in this model, called Base Fully Justified Representation (BFJR)
- BFJR can always be satisfied
- Stable priceable outcomes don't work here

Weighted Candidates

- An extension where candidates can have different weights for restriction purposes
- Can be used to model participatory budgeting
- A weighted counterpart of other proportionality axioms exists
- PAV fails completely
- Methods with priceable outcomes offer good approximations