Extending centrality measures to groups

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Preexisting knowledge

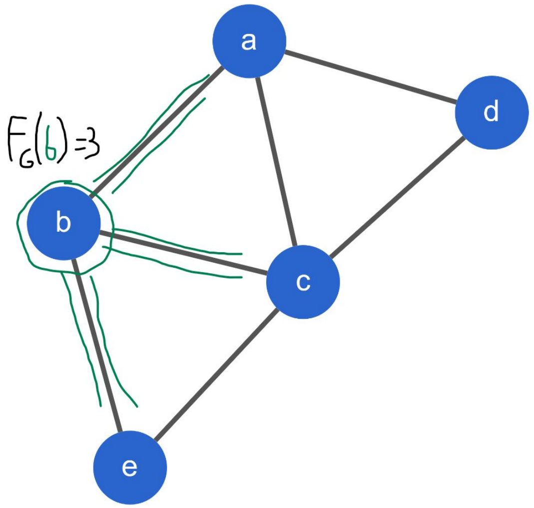
Earlier known methods of creating group centrality measures out of node centrality measures.

Centrality measures

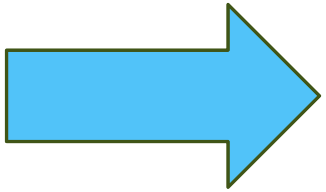
NODE CENTRALITY MEASURES

$F': G \times V \rightarrow \mathbb{R}$

Denoted as $F'_G(v)$



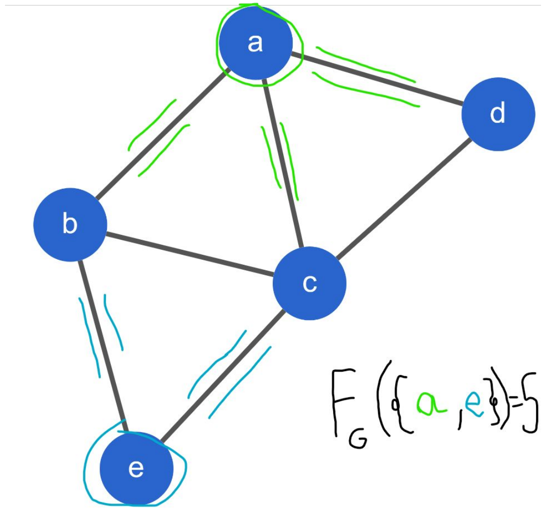
For graph $G = (V, E)$



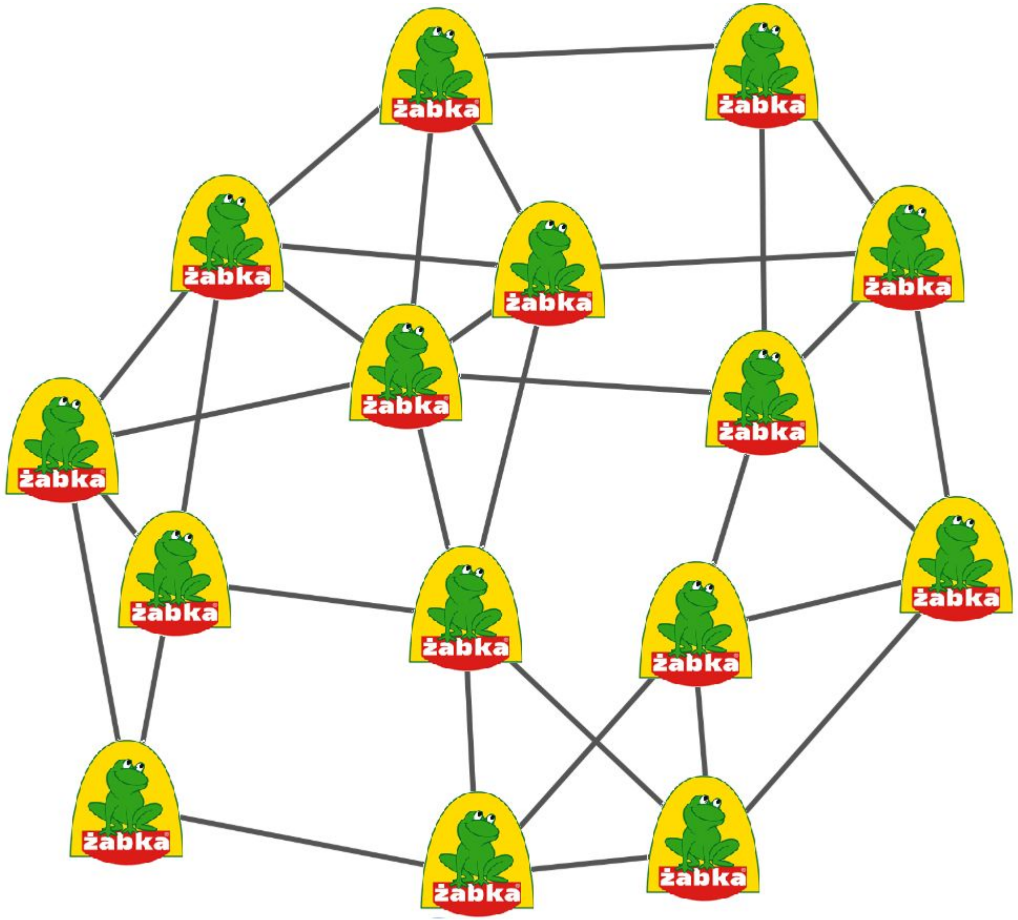
GROUP CENTRALITY MEASURES

$F: G \times 2^V \rightarrow \mathbb{R}$

Denoted as $F_G(S)$

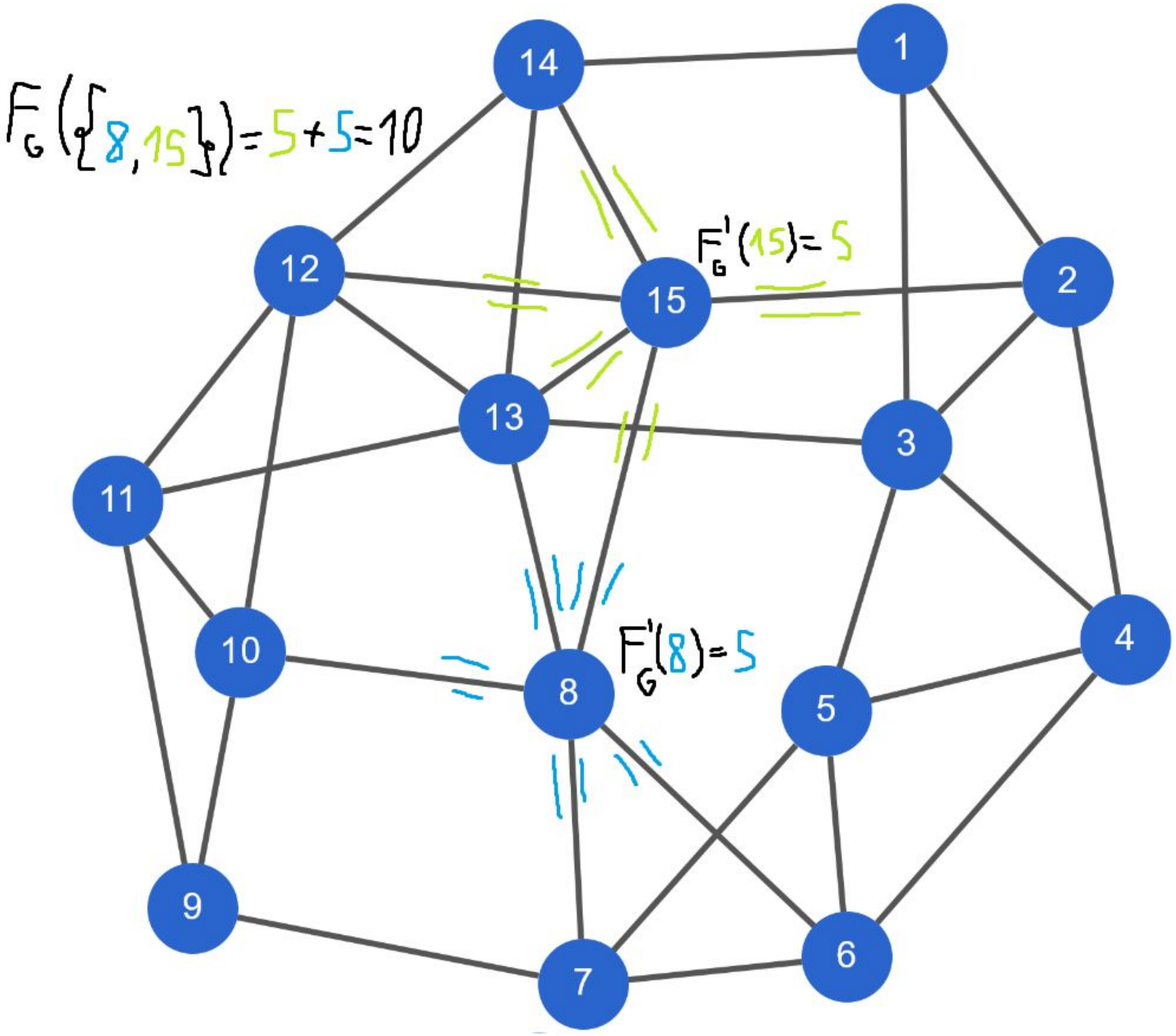


żabka ?



Sum-approach

$$F_G(S) = \sum_{v \in S} {F'}_G(v)$$



$$F_G(S) = \sum_{v \in S} F'_G(v)$$

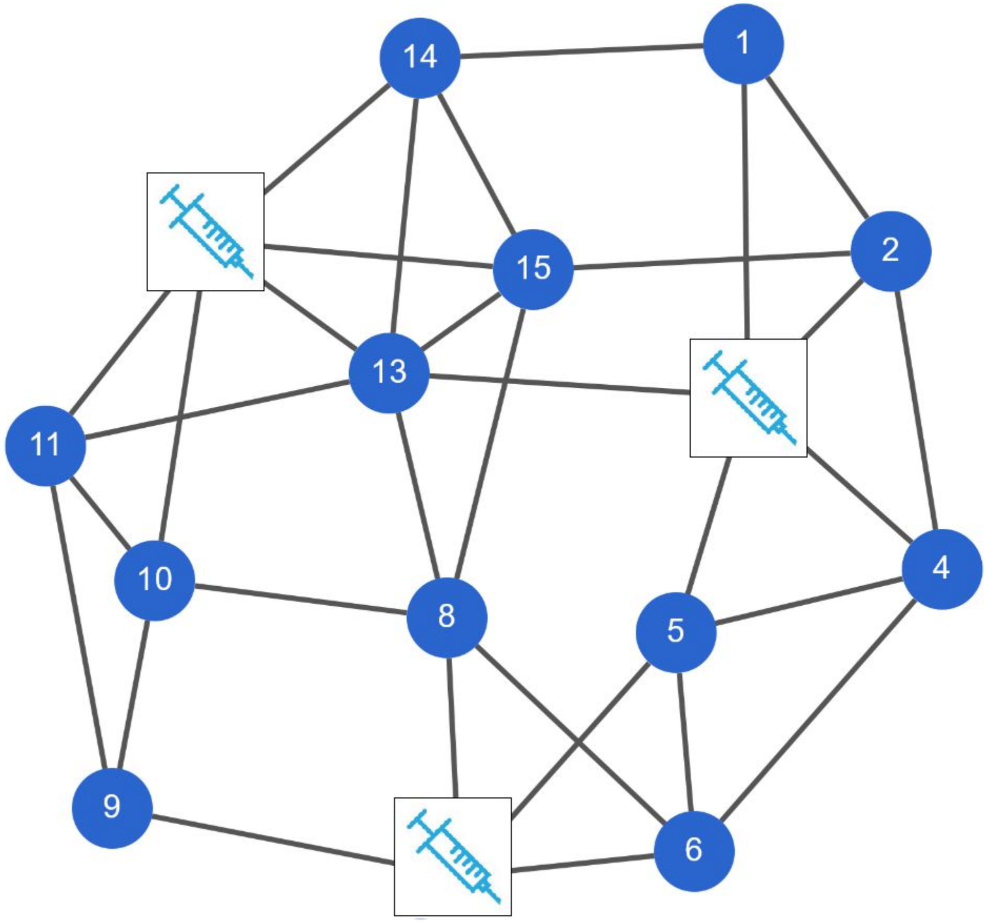
 x3 ?

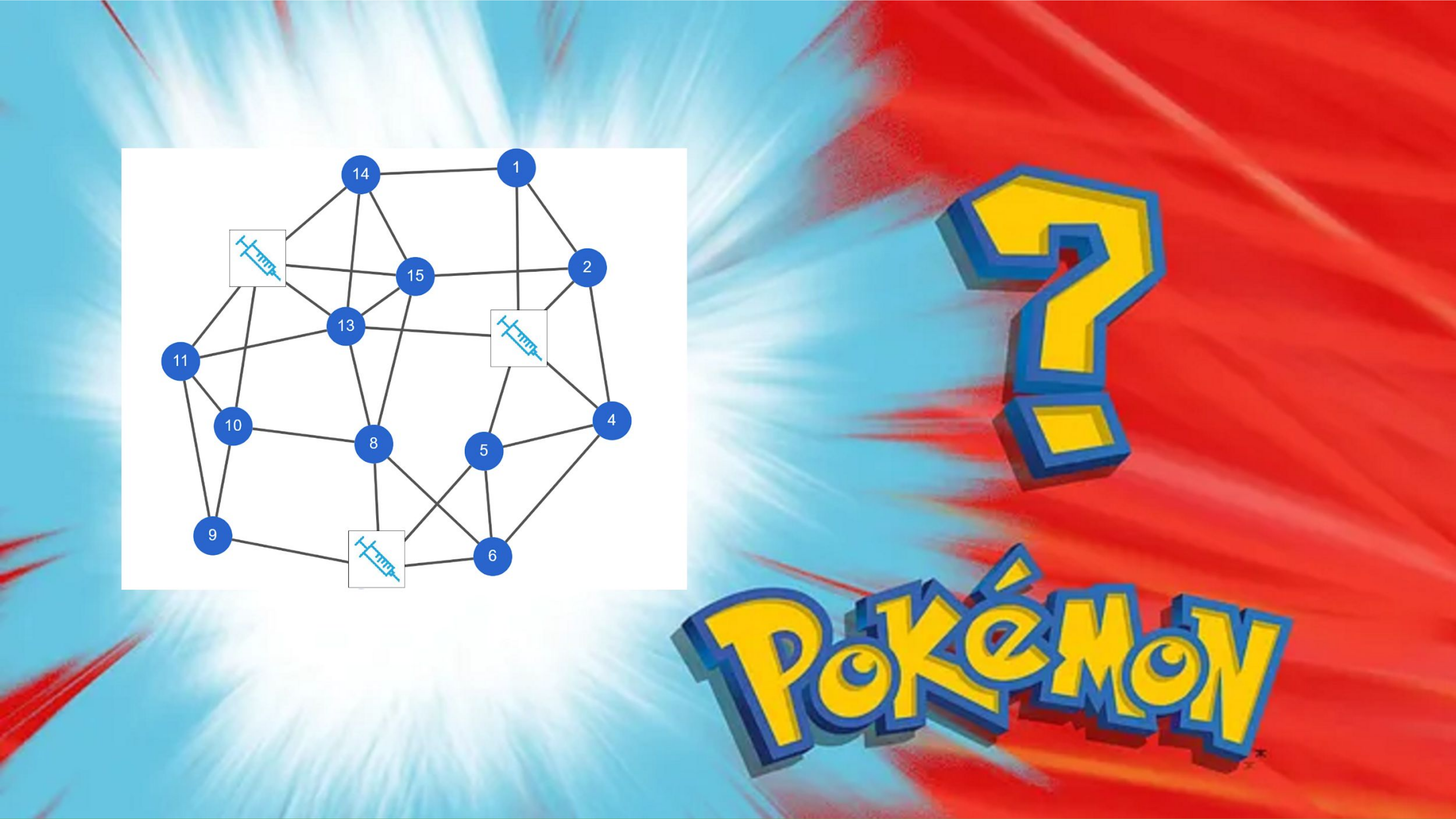
Node centrality measure: Closeness

$$F'_G(v) = \frac{1}{\sum_{u \in V} dist(u, v)}$$

Sum-approach gives unsatisfactory results

Could we achieve something like this?




14
1
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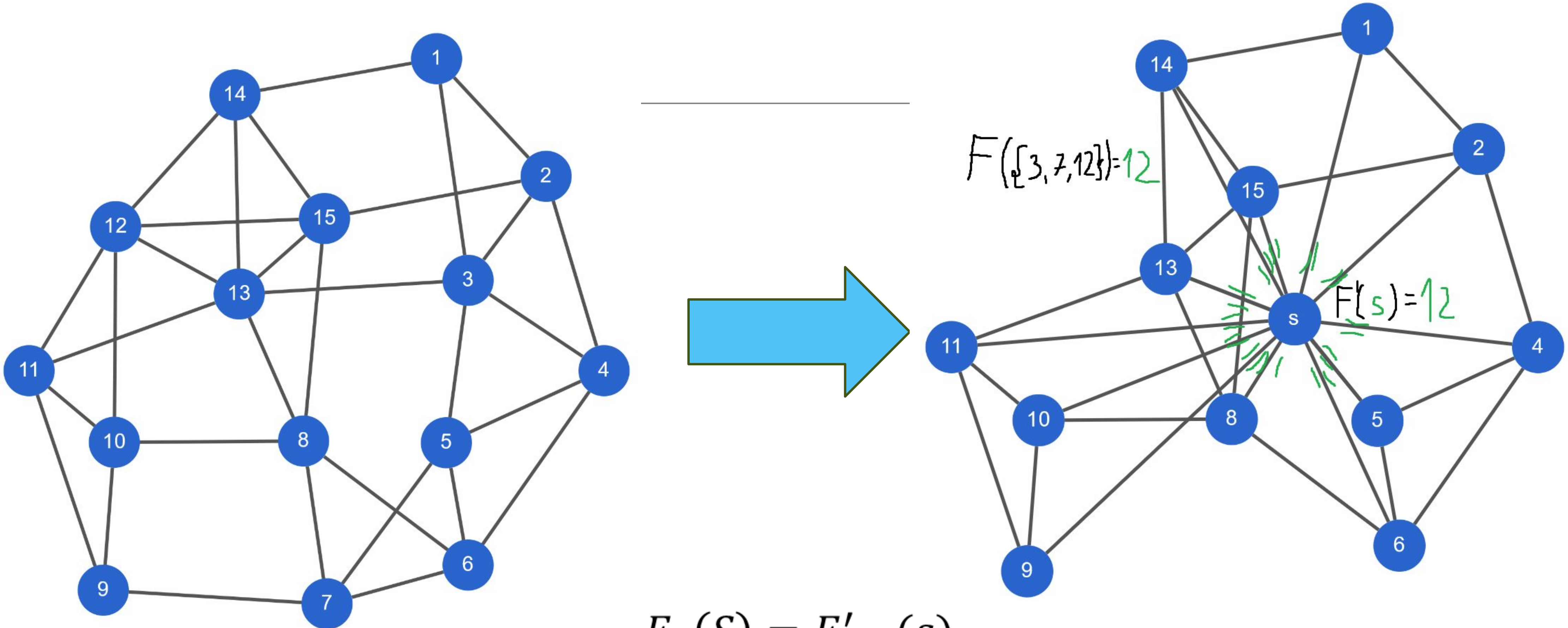
Merge-approach

$$F_G(S) = F'_{G^*}(s)$$

G^* is a graph with nodes from the group are merged into one

s is the node created by merging nodes from the group.



$$F_G(S) = F'_{G^*}(s)$$

G^* is a graph with nodes from the group are merged into one

s is the node created by merging nodes from the group.

x2

Node centrality measure: Betweenness

$$F'_G(v) = \sum_{s,t \in V \setminus \{v\}} \frac{|p \in \Pi_s(s,t) : v \in p|}{|\Pi_s(s,t)|}$$

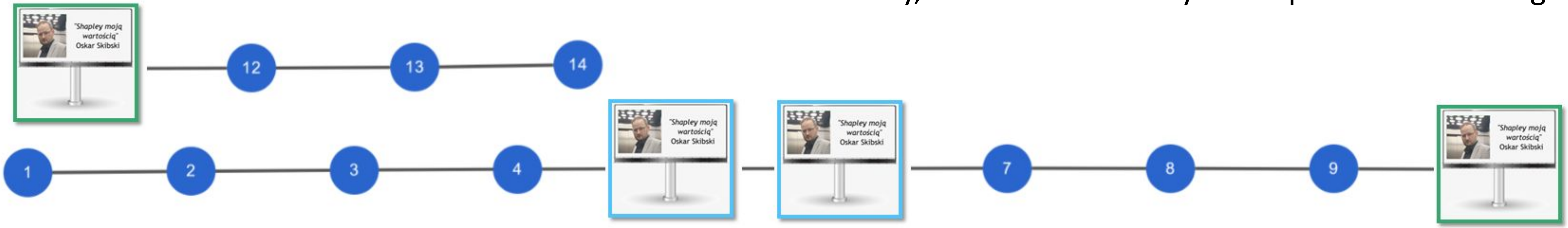
Where $\Pi_s(s,t)$ is the set of shortest paths from s to t.

Which arrangement is better - green or blue?

Merge-approach: green > blue.

Sum-approach (and me): this makes no sense.

ChatGPT: I'm sorry, but I cannot fulfill your request. As an AI langua



Total-approach

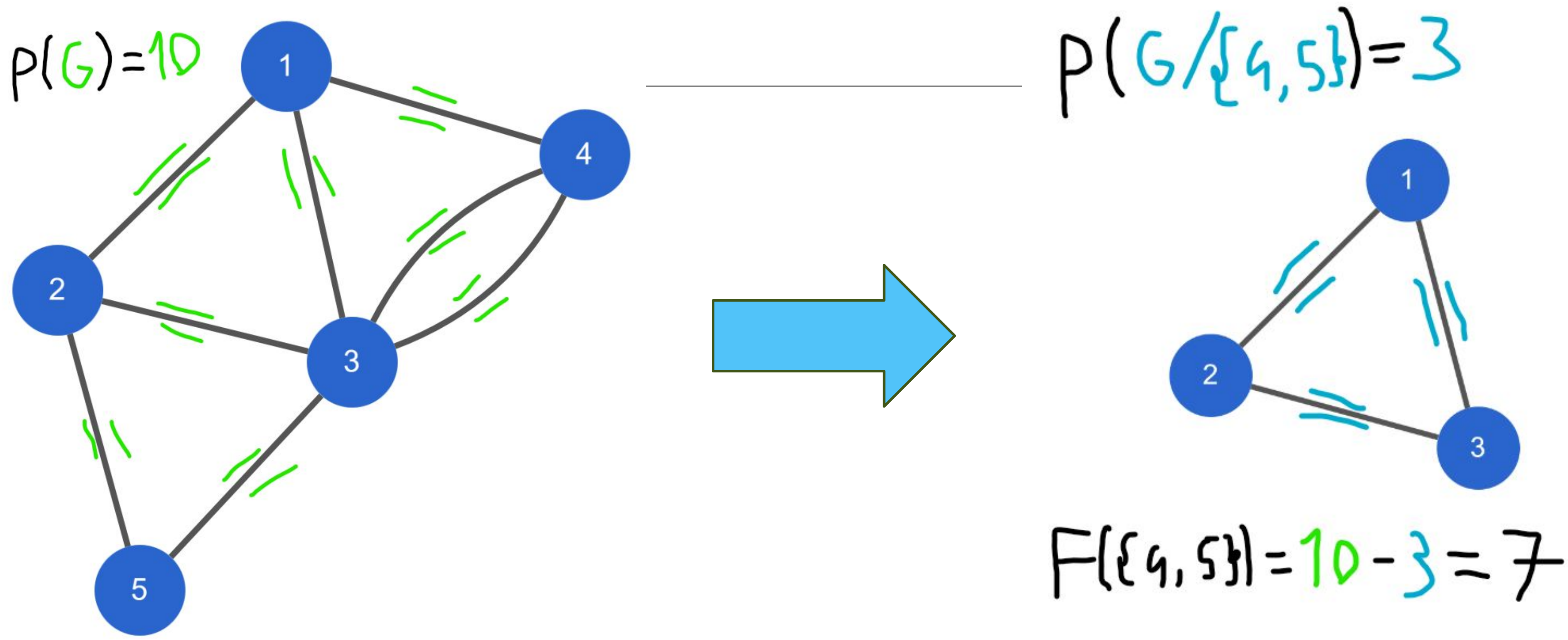
Only for vitality centralities!

Vitality centralities:

$\exists_{p:G\to\mathbb{R}}\forall_G \; {F'}_G(v) = p(G) - p(G \setminus \{v\})$

Then

$F_G(S) = p(G) - p(G \setminus S)$

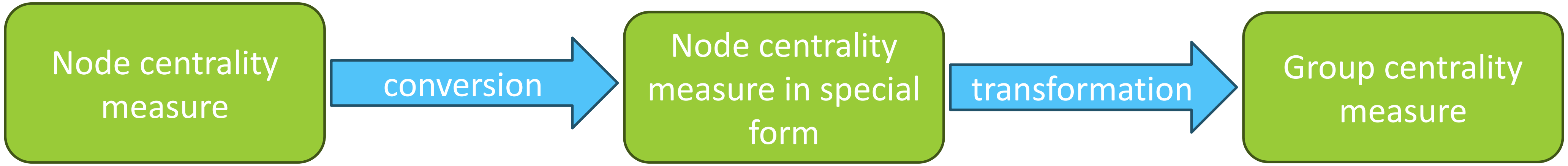


$$F_G(S) = p(G) - p(G \setminus S)$$

New idea

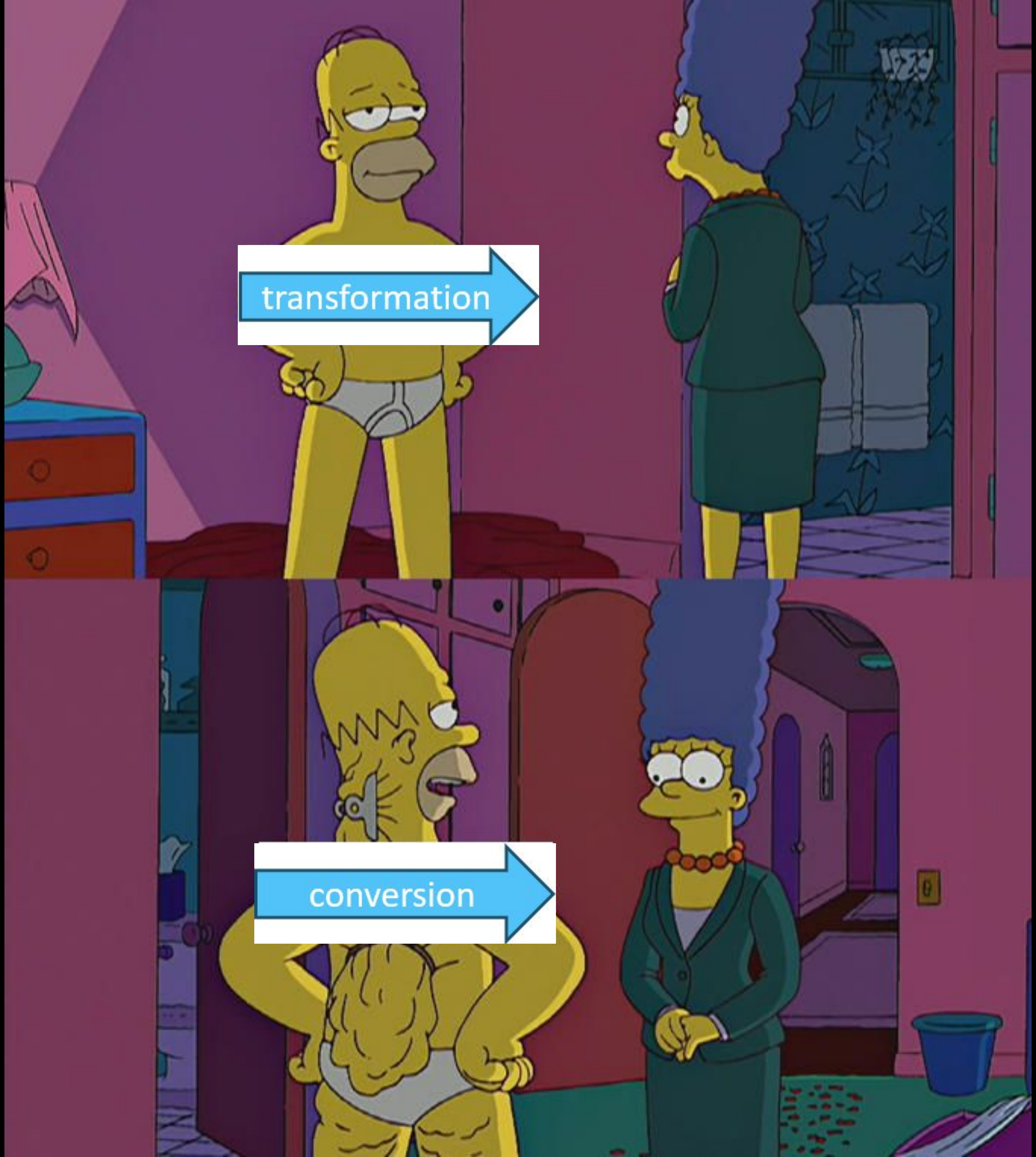
New method for extending node centrality measures to group centrality measures

The method



Conversion:
- Ambiguous
- Doesn’t always make sense
- Some choices have to be made

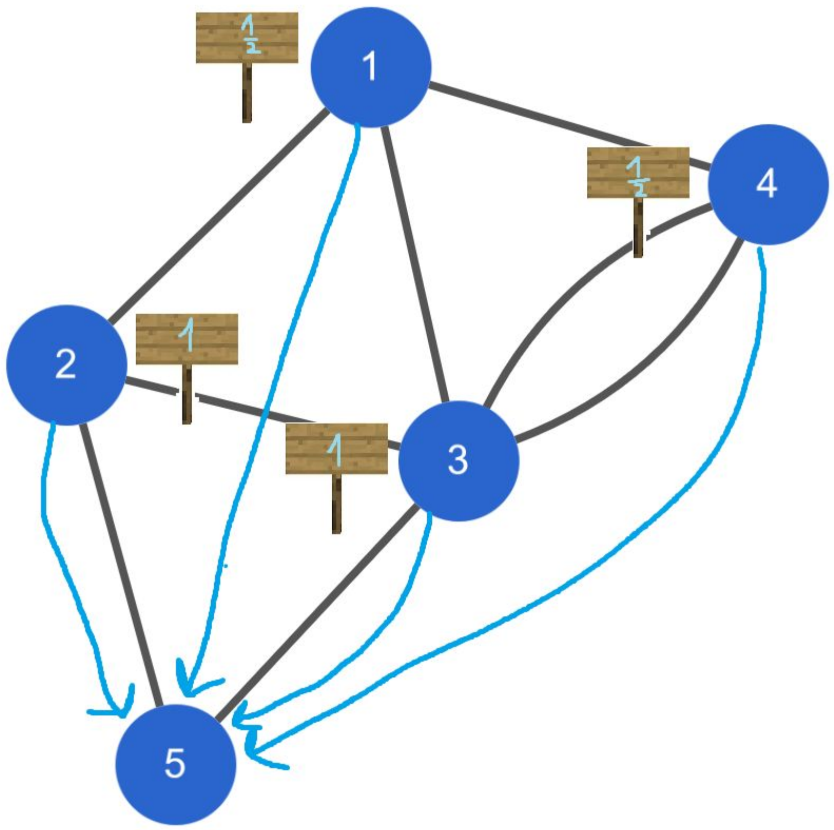
Transformation:
- Unambiguous
- Elegant
- Natural


transformation
conversion

Special form

Harmonic:

$F'_G(v) = \sum_{u \in V} \frac{1}{dist(u,v)}$



Special form*

$$F'_G(v) = \sum_{u \in V} f_G\,(u, v)$$

- Evaluation function $f: G \times V \times V \to \mathbb{R}$

Problems:

Betweenness

$$F'_G(v) = \sum_{s,t \in V \setminus \{v\}} \frac{|p \in \Pi_s(s,t): \mathrm{v} \in p|}{|\Pi_s(s,t)|}$$

Attachment

$$F'_G(v) = \sum_{U \subseteq V \setminus \{v\}} \frac{|U|!\,(|V| - |U| - |1|)!}{|V|!}\; 2(|K(G[U])| - |K(G[U \cup \{v\}])| + 1)$$

*almost

Special form*

$$F'_G(v) = \sum_{U \in \mathcal{F}^W} f_G(U, v)$$

- Evaluation function $f: G \times 2^V \times V \to \mathbb{R}$
- Evaluating set of nodes $\mathcal{F}^W \subseteq 2^W, W \subseteq V$

Problems:

Closeness

$$F'_G(v) = \frac{1}{\sum_{u \in V} dist(u, v)}$$

*almost

Special form

$$F'_G(v) = g(\sum_{U \in \mathcal{F}^W} f_G(U, v))$$

- Evaluation function $f: G \times 2^V \times V \to \mathbb{R}$
- Evaluating set of nodes $\mathcal{F}^W \subseteq 2^W, W \subseteq V$
- Normalization function $g: \mathbb{R} \to \mathbb{R}$, strictly increasing

Note that the normalization function does not change the group rankings

Transformation

$$F'_G(v) = g(\sum_{U \in \mathcal{F}^W} f(U,v))$$

sum

$$F_G(S) = g(\sum_{U \in \mathcal{F}^W} \sum_{v \in S} f_G(U,v))$$

max

$$F_G(S) = g(\sum_{U \in \mathcal{F}^W} \sum_{v \in S} f_G(U,v))$$

min

$$F_G(S) = g(\sum_{U \in \mathcal{F}^W} \sum_{v \in S} f_G(U,v))$$

proportional

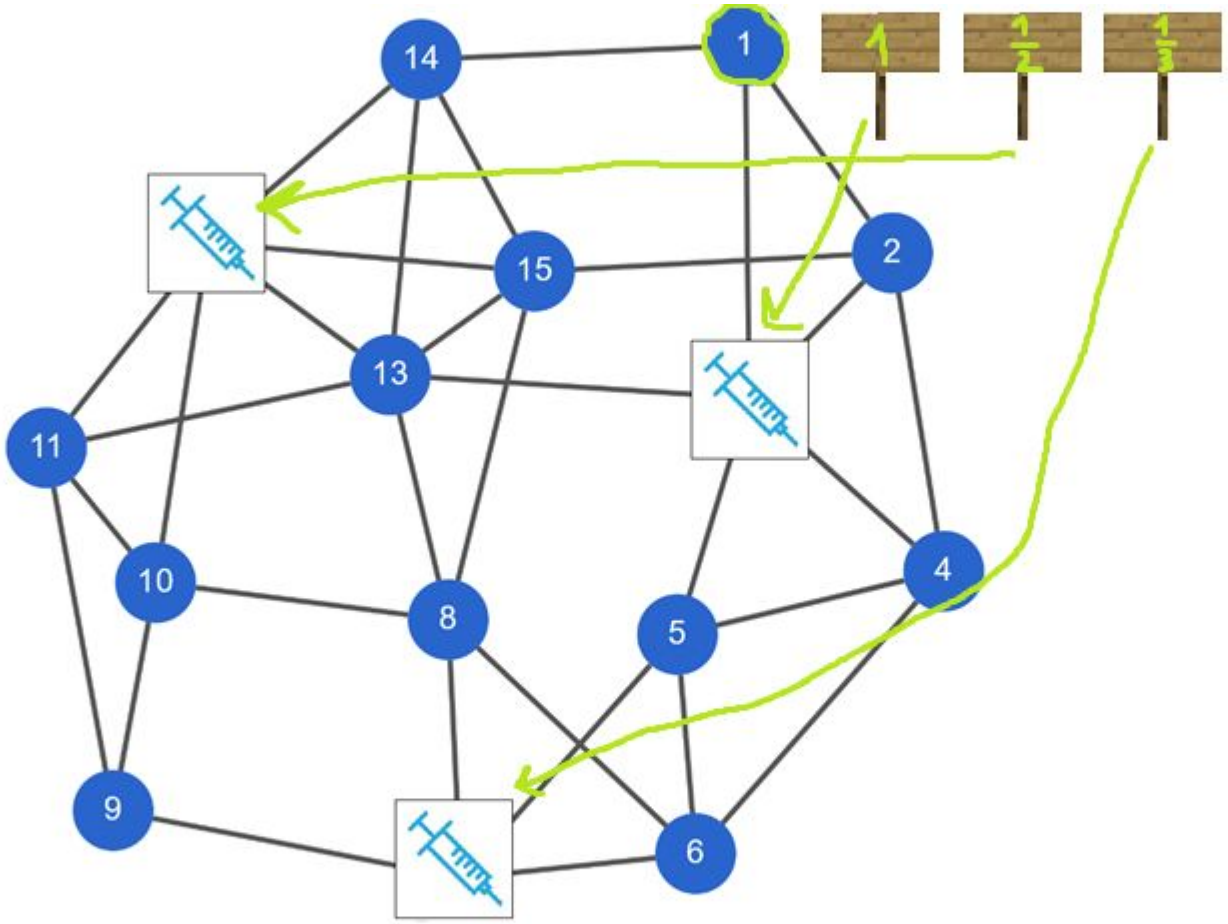
$$F_G(S) = g(\sum_{U \in \mathcal{F}^W} \sum_{v \in S} f_G(U,v))$$

$pav: \mathbb{R}^n \to \mathbb{R}$

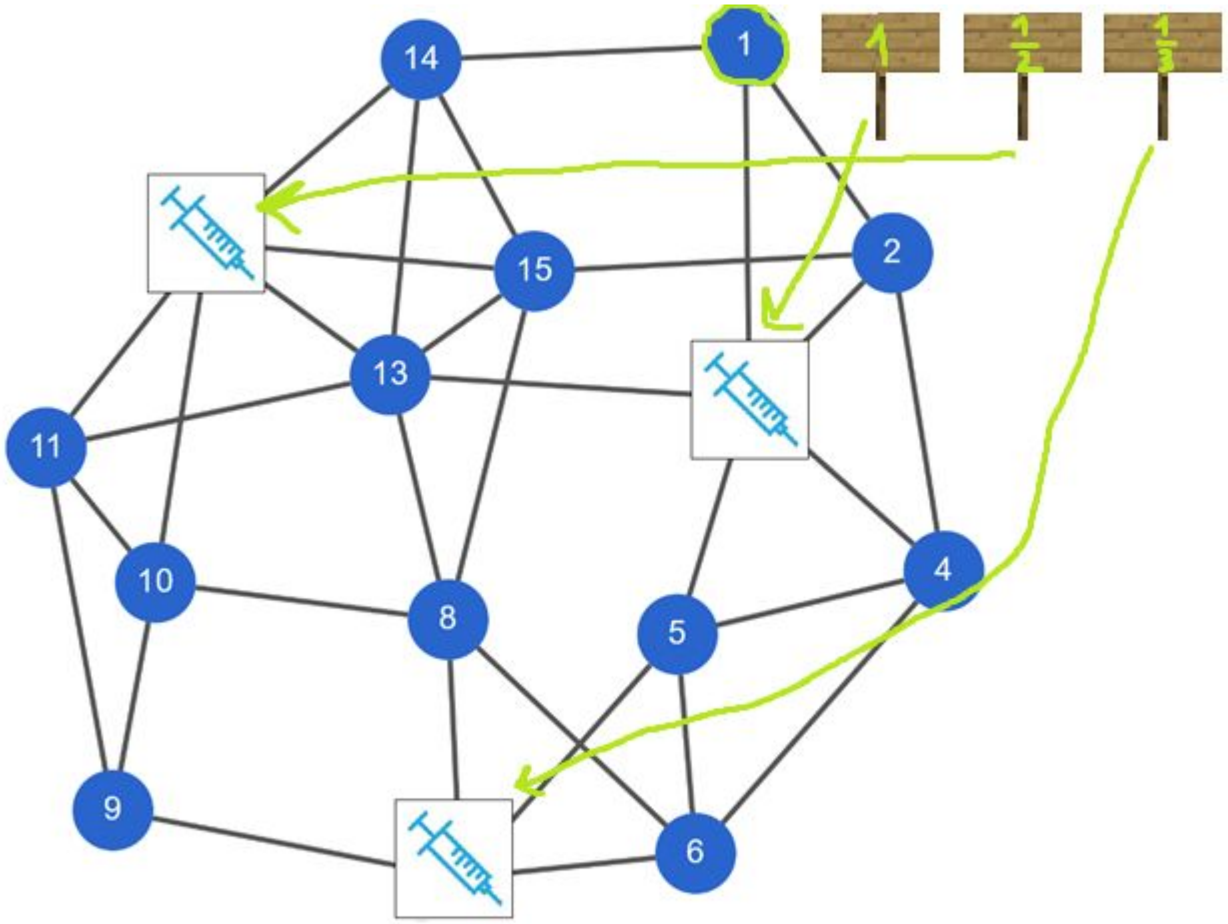
$pav(A_{1\ldots n}) = \sum_{i=1}^{n} \frac{1}{i} A_i$

(A jest posortowane malejąco)

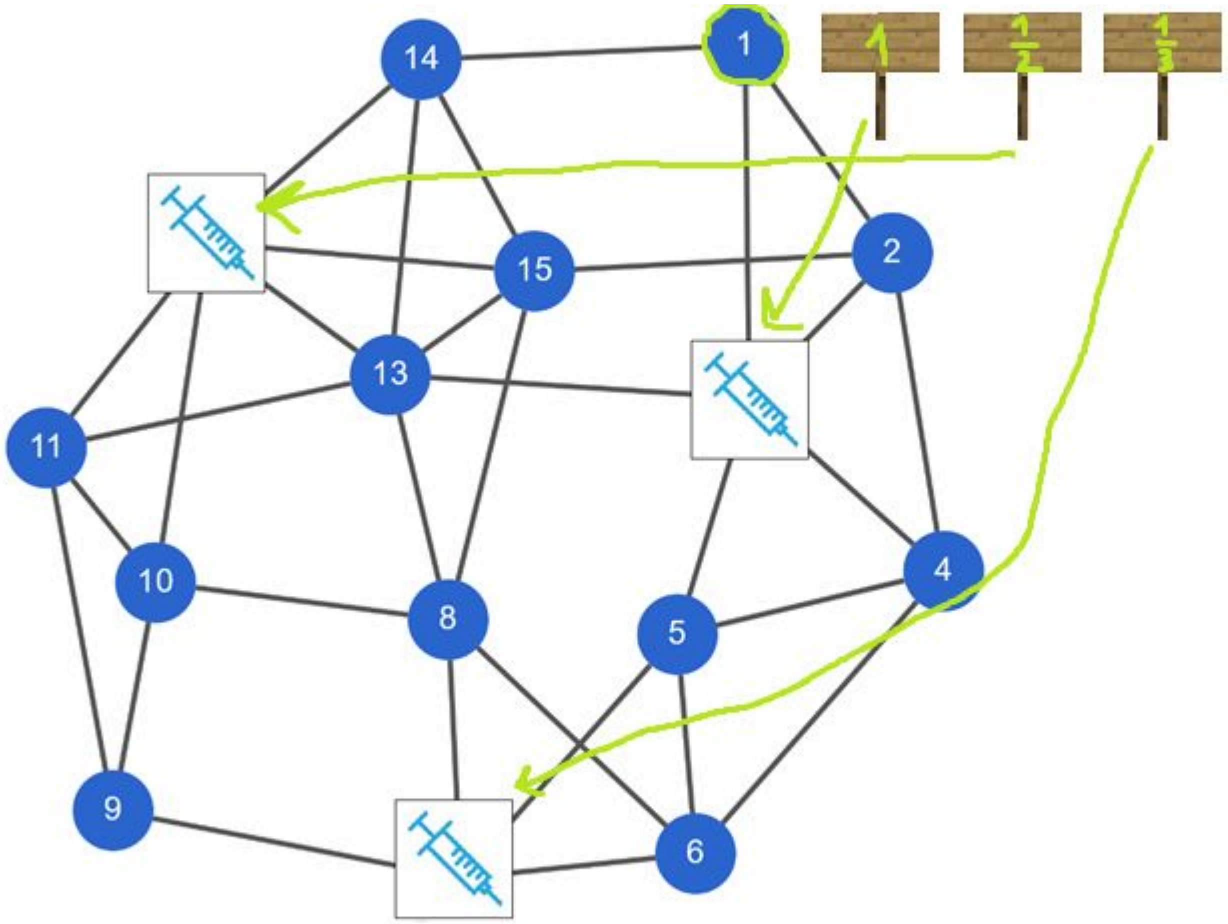
Intuition



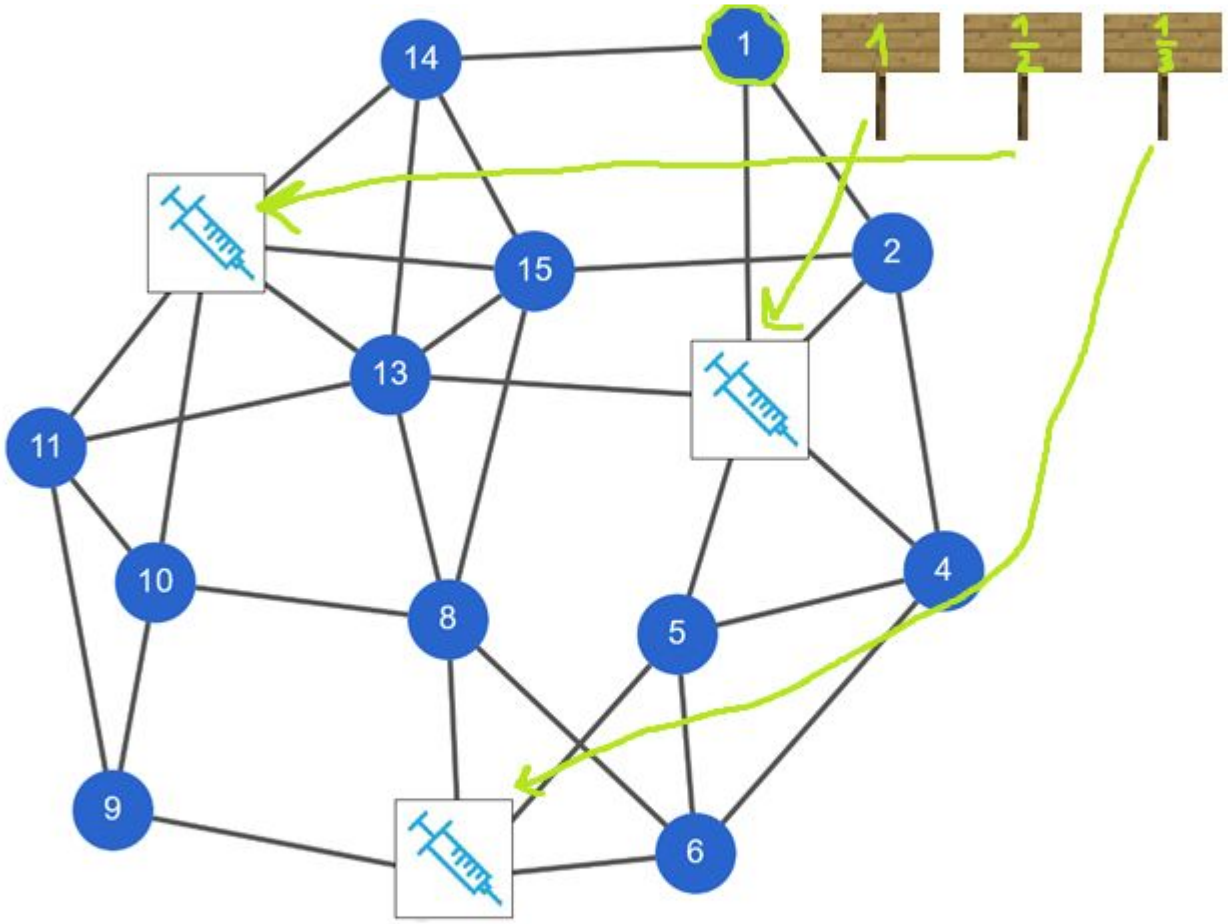
Intuition - sum



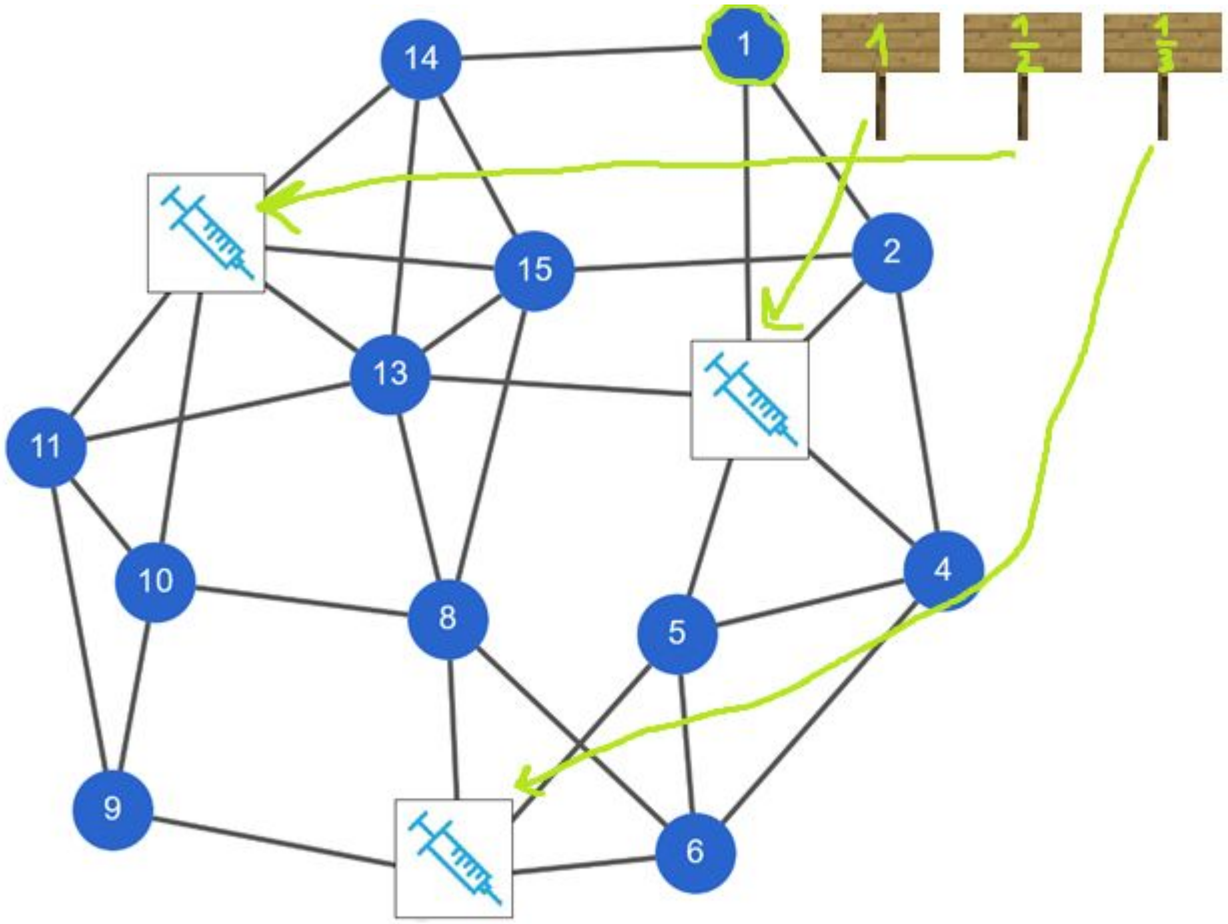
Intuition - max



Intuition - min



Intuition - proportional



Problem

$$CB_v(G) = \frac{1}{(n-1)(n-2)} \sum_{s,t \in V \setminus \{v\}} \frac{1}{2}\left(-|b_{s,t}(v)| + \sum_{e:v \in e} |x(e)|\right)$$

$$B_v(G) = \sum_{s,t \in V \setminus \{v\}} \frac{|p \in \Pi_s(s,t) : v \in p\}|}{|\Pi_s(s,t)|}$$

$$A_v(G) = \sum_{S \subseteq V \setminus \{v\}} \frac{|S|!(|V|-|S|-1)!}{|V|!} 2(|K(G[S])| - |K(G[S \cup \{v\}])| + 1)$$

$$C_v(G) = \sum_{u \in V \setminus \{v\}} \frac{1}{dist(u,v)}$$

$$\frac{1}{\sum_{u \in V \setminus \{v\}} dist(u,v)}$$

$$PR_v(G) = a \cdot \left(\sum_{u \in N_G(v)} \frac{PR_u(G)}{D_u(G)}\right) + b_v$$

$$Y_v(G) = \sum_{u \in V \setminus \{v\}} \delta^{dist(u,v)}$$

$$FB_v(G) = \sum_{s,t \in V \setminus \{v\}} flow_{s,t}(G) - flow_{s,t}(G[V \setminus v])$$

Possible solutions:

1. Nodes from the evaluated group don’t evaluate other nodes.
2. Nodes from the evaluated group don’t evaluate themselves.
3. Nodes from the evaluated group evaluate themselves.

A. Graph of apartment buildings and placing stores (Żabka) in them.
B. Modeling an infected population – which starting configuration of infected people will infect others the most.

Earlier known methods vs new method

Sum-approach	sum, V
Merge-approach	max, V \ S* (not always identical)
Total-approach	impossible
	sum, V \ S, $\mathcal{F}$ = {V\S}$***$ (very cheated)

Expressiveness - conversion

Eccentricity

$$C_v(G) = \frac{1}{\max_{u \in V} dist(v,u)}$$

$$F'_v(G) = g\left(\sum_{U \in \mathcal{F}^V} f_G(U, v)\right)$$

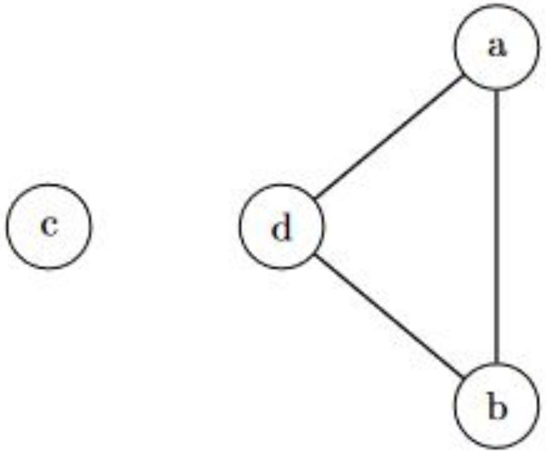
$$g(x) = x$$

$$f_G(U, v) = \begin{cases} F'_v(G) & U = \{v\} \\ 0 & \text{oth.} \end{cases}$$

$$\mathcal{F}^V = 2^V$$

Expressiveness – achievable group centralities (V)

For vitality measure:
$p(G) =$ [G has edges]



- min
  It cannot be expressed in our framework with min, since adding nodes to the group cannot make the measure bigger in the V variant, and $F_{\{a,b\}} > F_{\{a\}}(G)$
- sum
  Let's assume it's possible to express the group measure using a sum variant in our framework. Then

$$g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a)\right) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,c)\right) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a) + \sum_{U\in\mathcal{F}^V} f_G(U,c)\right) = 0$$

  Since g is increasing it implies

$$\sum_{U\in\mathcal{F}^V} f_G(U,a) = \sum_{U\in\mathcal{F}^V} f_G(U,c) = 0, g(0) = 0$$

  For analogous reasons

$$\sum_{U\in\mathcal{F}^V} f_G(U,b) = \sum_{U\in\mathcal{F}^V} f_G(U,c) = 0$$

  But then

$$F_{\{a,b\}}(G) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a) + f_G(U,b)\right) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a) + \sum_{U\in\mathcal{F}^V} f_G(U,b)\right) = g(0+0) = 0 \neq 1$$

  Which is a contradiction.
- max
  Let's assume it's possible to express the group measure using a max variant in our framework. Then

$$g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a)\right) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,c)\right) = g\left(\sum_{U\in\mathcal{F}^V} \max(f_G(U,a), f_G(U,c))\right) = 0$$

  This implies

$$\forall_{U\in\mathcal{F}^V} f_G(U,a) = f_G(U,c)$$

  For analogous reasons

$$\forall_{U\in\mathcal{F}^V} f_G(U,b) = f_G(U,c)$$

  Which implies

$$\forall_{U\in\mathcal{F}^V} f_G(U,a) = f_G(U,b)$$

  Which implies

$$F_{\{a,b\}}(G) = g\left(\sum_{U\in\mathcal{F}^V} \max(f_G(U,a), f_G(U,b))\right) = g\left(\sum_{U\in\mathcal{F}^V} f_G(U,a)\right) = 0 \neq 1$$

  Which is a contradiction

Expressiveness – achievable group centralities (V \ S)

$$g\left(\sum_{U\in\mathcal{F}^{V\setminus S}}\sum_{v\in S} f_G(U,v)\right)$$

$$g(x) = x$$

$$f_G(U,v) = \frac{F_{V\setminus U}(G)}{|V\setminus U|}$$

$$\mathcal{F}^W = \{W\}$$

$$\sum_{U\in\mathcal{F}^{V\setminus S}}\sum_{v\in S} f_G(U,v) = \sum_{v\in S} f_G(V\setminus S, v) = \sum_{v\in S}\frac{F_{V\setminus(V\setminus S)}(G)}{|V\setminus(V\setminus S)|} = \sum_{v\in S}\frac{F_S(G)}{|S|} = |S|\frac{F_S(G)}{|S|} = F_S(G)$$

$$F'_G(v) = g(\sum_{U \in \mathcal{F}^W} f_G(U, v))$$

Converting centrality classes – medial

LOOSE DEFINITION

A centrality measure is a *medial centrality* if there exists a function $\Delta_v^{s,t}(G)$ that evaluates the role of v in connecting s, t in G such that

$$F_v(G) = \sum_{s,t \in V} \Delta_v^{s,t}(G) \, .$$

Betweenness

$$B_v(G) = \sum_{s,t \in V \setminus \{v\}} \frac{|p \in \Pi_s(s,t) : v \in p\}|}{|\Pi_s(s,t)|}$$

Stress

$$S_v(G) = \sum_{s,t \in V \setminus \{v\}} |p \in \Pi_s(s,t) : v \in p|$$

Flow Betweenness

$$FB_v(G) = \sum_{s,t \in V \setminus \{v\}} flow_{s,t}(G) - flow_{s,t}(G[V \setminus v]) \, ,$$

Converting centrality classes– distance-based

$$F'_G(v) = g\left(\sum_{U \in \mathcal{F}^W} f_G(U, v)\right)$$

DEFINITION

A centrality measure is a *distance-based centrality* if there exists a function $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$F_v(G) = f\left(\left(dist_{u,v}(G)\right)_{u \in V}\right)$$

Distance-based Centralities

$$\sum_{u \in V \setminus \{v\}} a_{dist_{u,v}(G)}$$

Alternatives:

ADDITIVE (Harmonic, Decay, Degree, k-Step Reach)

		a_1	a_2	a_3	$a_{i>3}$
Harmonic [Rochat 2009]	$H_v(G) = \sum_u 1/dist_{u,v}(G)$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{i}$
Decay [Jackson 2005]	$Y_v(G) = \sum_u \delta^{dist_{u,v}(G)}$ where $\delta \in (0,1)$	δ^0	δ^1	δ^2	δ^i
Degree	$D_v(G) = \lvert\{u \in V \setminus \{v\} : (u,v) \in V\}\rvert$	1	0	0	0
k-Step Reach	$R_v^k(G) = \left\lvert\{u \in V \setminus \{v\} : dist_{u,v}(G) \le k\}\right\rvert$	1	1	1	$[i \le k]$
Eccentricity	$EC_v(G) = \max_{u \in V}\{dist_{u,v}(G)\}$				

Converting centrality classes – distance-based

$$F'_G(v) = g\left(\sum_{U \in \mathcal{F}^W} f_G(U, v)\right)$$

DEFINITION

A centrality measure is a *distance-based centrality* if there exists a function $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$F_v(G) = f\left(\left(dist_{u,v}(G)\right)_{u \in V}\right)$$

$$\sum_{u \in V \setminus \{v\}} a_{dist_{u,v}(G)}$$

- Closeness

$$C_v(G) = \frac{1}{\sum_u dist_{u,v}(G)}$$

$$F'_G(v) = g(\sum_{U \in \mathcal{F}^W} f_G(U, v))$$

Converting centrality classes – vitality

DEFINITION

A centrality measure is a *vitality index* if there exists a function $f: \mathcal{G} \to \mathbb{R}$ such that

$$F_v(G) = f(G) - f(G - v)$$

THEOREM

A centrality measure is a vitality index if and only if it is a Shapley-value based induced game-theoretic centralities.

$$F'_G(v) = g(\sum_{U \in \mathcal{F}^W} f_G(U,v))$$

Converting centrality classes – vitality

DEFINITION

A centrality measure is a *vitality index* if there exists a function $f: \mathcal{G} \to \mathbb{R}$ such that

$$F_v(G) = f(G) - f(G - v)$$

$$\sum_{U \subseteq V, U \neq \emptyset} f(U,v))$$

$$f(U,v) = \frac{(|U|-1)!(|V|-|U|)!}{|V|!}(F_\Sigma(U) - F_\Sigma(U \setminus \{v\}))$$

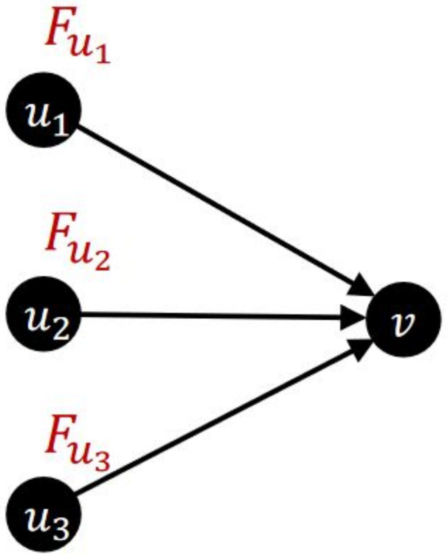
$F_\Sigma(U)$ is the sum of single node centralities of all nodes in the subgraph induced by U.

Converting centrality classes – feedback

Assess a node by the importance of its neighbors (direct predecessors).

LOOSE DEFINITION

A centrality measure is a *feedback centrality* if the centrality of a node (mostly) depends on the centralities and out-degrees of its predecessors.



Converting centrality classes – feedback

Recursive definition:

$$PR^a_v(G) = a\left(\sum_{(u,v)\in\Gamma^-_v(G)} \frac{c(u,v)}{\deg^+_u(G)} PR^a_u(G)\right) + b(v).$$

An Axiom System for Feedback Centralities

Walk definition:

$$p^a_{v,G}(t) = \sum_{\omega\in\Omega_t(G):\omega(t)=v} b(\omega(0)) \cdot \prod_{i=0}^{t-1} \frac{a \cdot c(\omega(i), \omega(i+1))}{\deg^+_{\omega(i)}(G)}$$

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$$PR^a_v(G) = \sum_{t=0}^{\infty} p^a_{v,G}(t)$$

Converting centrality classes – feedback

$$p^{a,h}_{v,u,G}(t) = \sum_{\omega \in \Omega_t(G):\omega(0)=u \wedge \omega(t)=v} b(u) \cdot \prod_{i=0}^{t-1} \frac{a \cdot c(\omega(i), \omega(i+1))}{h(\omega(i), G)}$$

$$h_c(v,G) = 1, h_d(v,G) = deg^+_v(G)$$

$$F(v) = \sum_{u \in V} f(u,v))$$

- PageRank $f(u,v) = \sum_{t=0}^{\infty} p^{a,h_d}_{v,u,G}(t)$, where $0 \leq a < 1$ is the PageRank parameter.
- Seeley index $f(u,v) = \lim_T \to \infty \sum_{t=0}^{T} \frac{p^{1,h_d}_{v,u,G}}{T}$
- Katz $f(u,v) = \sum_{t=0}^{\infty} p^{a,h_c}_{v,u,G}(t)$, where $0 \leq a < \frac{1}{\lambda}$ is the Katz parameter.
- Eigenvector $f(u,v) = \lim_T \to \infty \sum_{t=0}^{T} \frac{p^{1,h_c}_{v,u,G}}{T}$

Sources:

Axioms4Centalities

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