NP-hard metrics for evaluating PB election rules

Grzegorz Nowakowski

Preliminaries

- $E = (P, N, b, cost)$ – election
- $P = \{p_1, \ldots, p_m\}$ – projects
- $N = \{1, 2, \ldots, n\}$ – voters
- $b \in \mathbb{N}$ – budget
- $cost : P \to \mathbb{N}$
- $s_i(p) \in \mathbb{N}$ – score

Score utilities

$$u_i^{sc}(W) = \sum_{p \in W} s_i(p)$$

Cost utilities

$$u_i^{cost}(W) = \sum_{p \in W} s_i(p) cost(p)$$

Voting rules

voting rule – function $E \to P(P)$

$W \in P(P)$ – outcome, s.t. $cost(W) \leq b$

Outcome W is **exhaustive** if for each project $c \in C \setminus W$ we have that $cost(W) + cost(c) > b$

A voting rule is **exhaustive** if it always returns an exhaustive outcome.

Two popular voting rules are:

- Utilitarian Greedy,
- Method of Equal Shares.

Utilitarian Greedy

1. We start with an empty outcome $W = \emptyset$.
2. Repeatedly select a project p maximising the ratio $\sum_{i \in N} \frac{u_i(p)}{cost(p)}$.
3. If $cost(W) + cost(p) \leq b$ then add project p to W; otherwise remove the project from consideration.

This rule aims at maximizing the total utility of voters.

It is **optimal up to one project** – for each outcome W returned by UG there exists $p \notin W$ s.t.:

$$\sum_{i \in N} u_i(W \cup \{p\}) \geq \max_{W' : cost(W') \leq b} \sum_{i \in N} u_i(W').$$

Basic Metrics for Fairness and Efficiency

Average utility:

$$\frac{1}{n}\sum_{i\in N} u_i(W)$$

Dominance margin of R_1over R_2: the fraction of voters who enjoy strictly higher utility from the outcome of R_1than from R_2.

Similar ones: **improvement margin** and **exclusion ratio**.

Let's say, for a given outcome W, voter i's share is:

$$share_i(W) = \sum_{p\in W} \frac{s_i(p)}{\sum_{j\in N} s_j(p)} \cdot cost(p)$$

Power inequality:

$$\frac{1}{n} \cdot \sum_{i\in N} \left| share_i(W) - \frac{b}{n} \right| \cdot \frac{n}{b}$$

Basic Metrics for Fairness and Efficiency

Let $\mathcal{D} = \{D_1, \ldots, D_t\}$ be the set of districts (a partition of $\mathcal{N}$). The **dispersion of the budget allocation**:

$$\frac{1}{|\mathcal{D}|} \cdot \sum_{D \in \mathcal{D}} \frac{|\sum_{i \in D} share_i(W) - |D|/n \cdot b|}{|D|/n \cdot b}$$

City	Add1U, C	Util. G, D	Util. G, C
Czestochowa	0.23	0.28	0.39
Gdansk	0.27	0.33	0.46
Katowice	0.19	0.26	0.51
Krakow	0.08	0.24	0.23
Warsaw	0.20	0.41	0.41
Wroclaw	0.15	0.26	0.22
Zabrze	0.38	1.24	0.41

Let W_{sc} and W_{appr} be the outcomes of a given voting rule for the original and the approval elections, respectively. The **robustness ratio**:

$$\frac{cost(W_{appr} \cap W_{sc})}{cost(W_{sc})}$$

City	Add1U, C	Util. G, D	Util. G, C
Czestochowa	0.80	0.35	0.39
Gdansk	0.87	0.26	0.39
Katowice	0.83	0.56	0.42
Krakow	0.78	0.52	0.41

Budget distribution among Categories

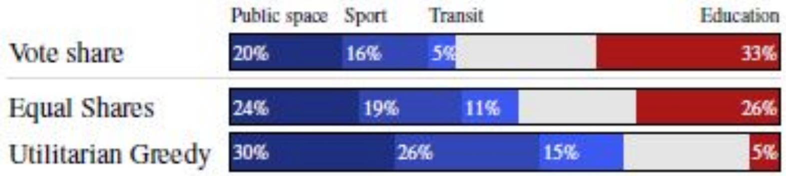
For each project p, denote by $tags(p)$ the tags assigned to p. For each tag t, we can compute **vote share**:

$$\frac{1}{n}\sum_{i\in N}\sum_{p\in A_i:t\in tags(p)}\frac{1}{|A_i|\cdot|tags(p)|}$$

and also for an outcome W we define **spending share**:

$$\frac{1}{cost(W)}\sum_{p\in W:t\in tags(p)}\frac{cost(p)}{|tags(p)|}$$

With these two, we can compute l_2 distance between the vectors of vote shares and the spending shares of all the tags.



The Core

For an election $(N, C, b, cost)$ we define **extended justified representation:**

$$\forall_E \forall_{l \in \mathbb{N}} \forall_{S \subseteq N : S \text{ is } l\text{-cohesive}} \exists_{i \in S} |A(i) \cap R(E)| \geq l$$

Committee W is in the **core** if

$$\forall_{S \subseteq N} \forall_{T \subseteq C} \left(\frac{|S|}{|N|} \geq \frac{cost(T)}{b} \Rightarrow \exists_{i \in S} u_i(W) \geq u_i(T) \right)$$

Is the core always non-empty?

What we know:

- Core can be empty in elections with ranking-based preferences and in PB with additive utilities
- $Core \implies EJR$

Pabutools and Master’s Thesis

- Pabutools – Python library with implementation of some rules for PB
- Implement metrics for evaluating elections using linear programming
- Research whether the core can be computed fast in some cases (maybe with additional assumptions / constraints)
- Possibly research other metrics such Fractional Core, Pareto-Optimality, etc.

Thanks for your attention