Attaining Equilibria Using Control Sets

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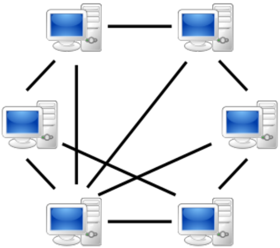
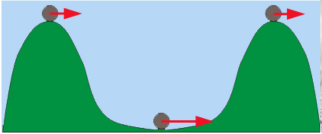


Aims

In an interaction (society, service providing, politics, etc.), while playing a profile, consider some equilibrium.

- Find the easiest way to motivate playing this equilibrium
- Find and motivate in reasonable time

We want to quickly move to the desired NE



Goals and Plan

Motivating equilibria is often external - by subsidies, taxes, mediation, etc.

Internal

Assume that some agents are motivated/bribed/controlled to act and thereby motivate the others to act as desired.

The plan:

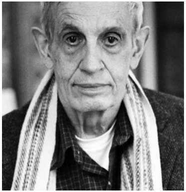
1. Model and comparison with Stackelberg and SNE
2. General games: hardness and algorithms
3. Potential games
    1. Coordination games on graphs
    2. Singleton congestion games
    3. Symmetric congestion games with decreasing costs
4. Conclusions and the future

Model

1. $G = (N, S = S_1 \times S_2 \times \ldots \times S_n, (u_i)_{i=1,\ldots,n})$,
2. The solutions, forming a solution set, are a set of strategy profiles $D \subseteq S$,
3. Our default is Nash equilibrium, though our definitions are general

$$\forall i \in N, \forall s_i' \in S_i : u_i(s) \geq u_i(s_i', s_{-i}), \tag{1}$$

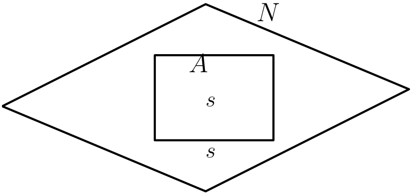
where $s_{-i} \stackrel{\Delta}{=} (s_1, \ldots, s_{i-1}, s_{i+1}, s_n)$.



Model – Direct Control Set

Definition (Direct Control Set)

Consider any profile s and a Nash eq. $d \in NE$. A set of agents $A \subseteq N$ can bring s to d directly (by playing d), while all the others play s, if for each agent outside A, d is a BR.
A is called a direct control set with respect to G, s, and d.
The minimum size of a direct control set is called the direct control number, denoted by $\mathrm{DirConNum}(G, s, d)$.



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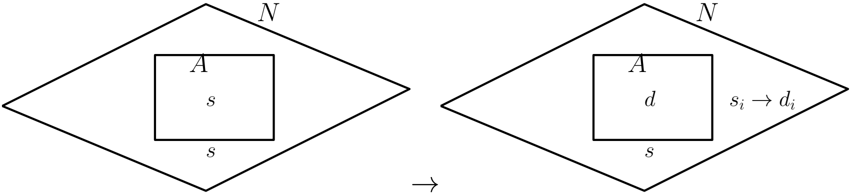
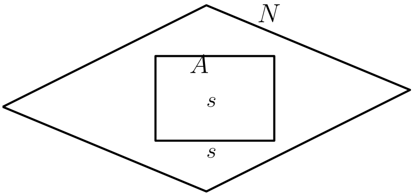


Model – Direct Control Set Problem

Definition (Direct Control Set Problem)

The *direct control set* *problem receives a game* $G = (N, S, (u_i)_{i=1,\ldots,n})$, *profile* s *and equilibrium* d, *and weights* $w \colon N \to \mathbb{R}_+$.
A *solution* *is a set of agents* $A \subseteq N$.
A *feasible solution* *is a solution that can bring* s *to* d *directly,*
Find a feasible solution A *with the minimum* $w(A) \triangleq \sum_{a \in A} w(a)$.



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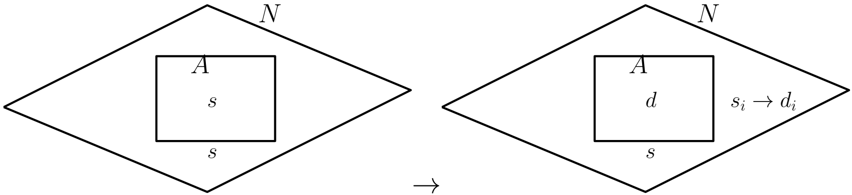


Model – Direct Control Set Problem - Representation

Definition (Direct Control Set Problem)

The direct control set problem receives a game $G = (N, S, (u_i)_{i=1,\ldots,n})$*, profile* s *and equilibrium* d*, and weights* $w \colon N \to \mathbb{R}_+$*.*
A solution is a set of agents $A \subseteq N$*.*
A feasible solution is a solution that can bring s *to* d *directly,*
Find a feasible solution A *with the minimum* $w(A) \triangleq \sum_{a \in A} w(a)$*.*

Remark (Representation)

Impracticable n *utility* n*-dimensional matrices would take* $\Omega(n \cdot 2^n)$ *space, rendering the brute-force algorithm polynomial.*
Therefore, we explicitly store the strategies, while the utilities are provided by a polynomial oracle.

Optimum Sizes - General Games

Proposition

For any natural numbers $n > p \geq 1$*, there exists a game* G *with* n *players, possessing equilibria* s *and* d*, such that* $\mathrm{DirConNum}(G, s, d) = p$*.*

Corollary

Can get up to $n - 1$*, but not more.*

Control vs Strong NE

Theorem

Assume that game G satisfies both following conditions for some $d \in NE$:

1. *Given any profile s, any subset of at least m agents is a direct control set with respect to G, s and d.*
2. *For any player j, playing d_j in d is at least as profitable for her as in any other profile.*

Then, d is an $n - m$-strong NE, and this is tight for every $m < n$.

The converse does **not** hold!

Hardness - Decision

Theorem

The decision versions of direct control set is NP-complete.

Proof.

NP-hardness holds even for coordination games on graphs, shown later. These problems belong to NP, the witness being a control set (poly). □

Hardness to Approximation

Theorem

Is not approximable within factor $n^{1-\epsilon}$, $\forall \epsilon > 0$, *unless* $P = NP$.
Holds even when bringing an NE to an NE, and each player has 2 strategies.

Proof.

Reduction from the problem of deleting a minimum vertex set from a connected graph $G = (V, E)$ to obtain a tree. □

We are going to cope by

- DP on trees
- Approximating subclasses assuming monotonicity w.r.t. inclusion
- Classes of potential games

Dynamic Programming (DP) Algorithms

Given a game, transform it to the graph where

1. The nodes are the players,
2. Two nodes are connected $\iff$ there is some influence

Now,

1. If the graph is a tree, then Dynamic Programming solves optimally.
2. If we can obtain such a graph by taking out $\leq c\text{OPT}$ weight from the game, then we obtain a $1 + c$-approximation, assuming monotonicity.

Monotonicity-Based Algorithms - Monotonicity

Definition

Monotonicity w.r.t. inclusion means that if A is a $\mathrm{DirConSet}$ *w.r.t.* (G, s, d), *then any including set* $A' \supset A$ *is also a* $\mathrm{DirConSet}$ *w.r.t.* (G, s, d).

Monotonicity does not generally hold, even in singleton congestion games, but we'll assume it by default.

We'll even assume monotonicity w.r.t. motivating each single player.

The idea: local ratio is extended inductively

$C \leftarrow \emptyset$

While $\exists$ a player $i \in N \setminus C$, s.t. $d_i \notin BR_i(d_C, s_{N \setminus C})$

1. Let $N[i] \leftarrow \{i\} \cup \{j \in N \setminus C : s_j \text{ can influence } u_i\}$
2. Set $\epsilon \leftarrow \min_{j \in N[i]} w(j)$
3. $w(j) \stackrel{\Delta}{=} w(j) - \begin{cases} \epsilon & \text{if } j \in N[i], \\ 0 & \text{otherwise} \end{cases}$
4. $C \leftarrow C \cup \{k \in N \colon w(k) = 0\}$

Since the optimum picks at least one and at most all of $N[i]$, the local ratio of the picked solution to the optimum is $\leq 1+$ the maximum intersection number.

$\Rightarrow$ also the total ratio.

Potential Games

Consider $\mathrm{DirConSet}$ in potential games. Recall

Definition

Potential games are strategic games there exists a function $P : S \to \mathbb{R}$*, where for any* $s_{-i} \in \times_{j \in N \setminus \{i\}} S_j$ *and for any strategies* s_i, t_i *of player* i*,*
$u_i(s_i, s_{-i}) - u_i(t_i, s_{-i}) = P(s_i, s_{-i}) - P(t_i, s_{-i})$.

Monderer and Shapley (1996) proved that the set of finite potential games is isomorphic to the set of congestion games, which is

Definition

A congestion game consists of $(N, \Sigma, (S_i)_{i \in N}, (c_r)_{r \in \Sigma})$*, where each* $S_i \subseteq 2^{\Sigma}$.
Each $r \in \Sigma$ *has an increasing cost function* $c_r \colon [n] \to \mathbb{R}$*, and*
$C_i(s) \triangleq \sum_{r \in s_i} c_r(l_r(s))$*, where* $l_r(s) \triangleq |\{j \in N : r \in s_j\}|$.

What are the possible sizes of $\mathrm{DirConSet}$ in congestion games?

Congestion Games - Optimum Sizes

First, recall that

Proposition

For any natural numbers $n > p \geq 1$, there exists a game G with n players, possessing equilibria s and d, such that $\mathrm{DirConNum}(G, s, d) = p$.

Interestingly, for congestion games, the upper bound does not decrease

Proposition

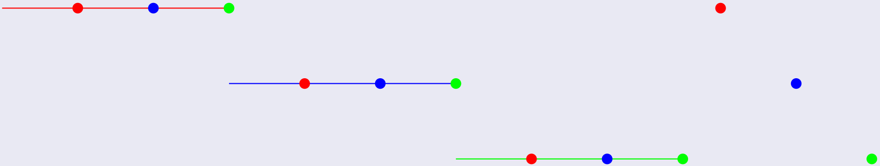
For any $n > p \geq \frac{n-1}{2}$, there exists a congestion game G with n players, possessing equilibria s and d, such that $\mathrm{DirConNum}(G, s, d) = p$.

Congestion Games - General Congestion Games

Proof Illustration for $n = 3$.

1. Player 1 either plays the red line, or all the red dots.
2. Player 2 either plays the blue line, or all the blue dots.
3. Player 3 either plays the green line, or all the green dots.

$\Rightarrow$ To motivate i to move from the “line”-NE to the “dots”-NE, at least p (depending on the costs) other players have to move first.



Proposition

For any $n > p \geq \frac{n-1}{2}$, there is a congestion game with $\mathrm{DirConNum}(G, s, d) = p$.

Proof.

For any n, let $N \triangleq [n]$ and $\Sigma \triangleq \left[n^2 + n\right]$, and define the costs: For any $e \in \left[n^2\right]$, let

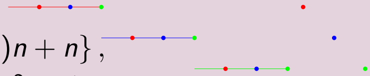
$$c_e(x) \triangleq \begin{cases} x & x \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

For any $e \in \left\{n^2 + 1, \ldots, n^2 + n\right\}$, let (for a fixed p)

$$c_e(x) \triangleq \begin{cases} 2(2p + 1 - n) \cdot x & x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

Now, let player i's strategy set be



$$S_i \triangleq \{\{(i-1)n+1, (i-1)n+2, \ldots, (i-1)n+n\},\ \{i, n+i, 2n+i, \ldots, (n-1)n+i\} \cup \{n^2+i\}\}.$$

Everyone playing the first strategy is an NE (no intersection), and we now prove that everyone playing the second one is an NE, too. Moving from the first NE to the second one requires at least p players to move. Indeed, when q players play the dots-strategies, keeping playing the lines-strategy costs $q \cdot 2$, while deviating to the dots-strategy would incur $(n-q-1) \cdot 2 + 2(2p+1-n) = 2(2p-q)$, and

$q \cdot 2 < 2(2p-q) \iff q < p$.

Substituting $q \leftarrow n-1$, implies everyone playing the second strategy is an NE. □

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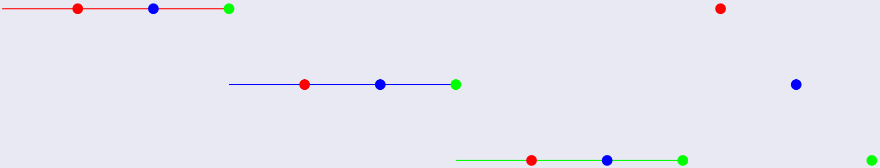


□

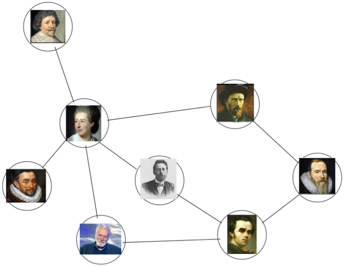
Subclasses of Potential Games

The congestion games are often hard even to approximate $\Rightarrow$ consider interesting subclasses!
Will first prove hardness.

Coordination Games - Model

Definition

A coordination game (Apt et al. 2017) is defined by an undirected graph $G = (N, E)$ *without self-loops, which nodes are players.*
Player i *selects a colour* $s_i \in S_i$ *of prestige* $p(s_i)$,
and $u_i(s) \triangleq p(s_i) \left|\{j \in \mathrm{N}(i) | s_i = s_j\}\right|$.
We assume $0, i \in S_i, \forall i$ *and the target equilibrium is* $d = (0, 0, \ldots, 0)$.



Coordination Games - Properties

Proposition

In a coordination game, being a direct control set is monotonic w.r.t. inclusion.

Observation

Coordination games are potential games, the potential being $P(s) \triangleq \sum_{e=(i,j) \in E} \xi(s_i, s_j) \cdot p(s_i)$, *where* $\xi(s_i, s_j)$ *is the indicator*

$$\xi(s_i, s_j) \triangleq \begin{cases} 1 & s_i = s_j, \\ 0 & \textit{otherwise}. \end{cases}$$

Coordination Games - Hardness

Theorem

Direct control set in a coordination game is NP-complete and not approximable within $\Omega(\log(n))$, even for equal prestige $(p(s_i))$.

Proof.

Given an instance of dominating set $G = (V = \{1, \ldots, n\}, E)$, define the following coordination game.

1. Make two copies of the original graph, where both copies of each vertex are connected among themselves and to all the copies of the neighbours of that vertex in the original graph.

Example

becomes

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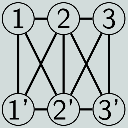

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1. Make two copies of the original graph, where both copies of each vertex are connected among themselves and to all the copies of the neighbours of that vertex in the original graph.
2. Let each copy be controlled by a separate player.
3. Let $S_i = \{0, i\}$.

Here, a direct control set from $s = (1, 2, \ldots, n)$ to $d = (0, 0, \ldots, 0) \iff$ projects a dominating set in the original graph. □

Coordination Games - Algorithms

Approximation when each $S_i = \{0, 1\}$

For approximation, reduce the problem to the weighted k-dominating set problem ($\triangleq$ need at least k neighbours from the chosen set in the neighbourhood of every non-chosen vertex.):

1. For each vertex i, let k_i be the minimum number of neighbours required for i to have 0 at a best response.
2. Let $k \triangleq \max_{i \in N} k_i$.
3. Add $k - k_i$ unique zero-weighted neighbours to each original vertex. Make each such vertex play 0. $//i$ requires $\geq k$, having $k - k_i$ for free

Congestion Games - Singleton

Let's zoom in into singleton congestion games with increasing costs.

Definition

A singleton congestion game is a congestion game, where $|s_i| = 1, \forall i \in N, s_i \in S_i$.

Theorem

For singleton congestion games with n *players and* m *resources, there always exists a* $\mathrm{DirConSet}$ *of size at most* $n - \lceil n/m \rceil$*, and this is tight.*

Proof.

When bringing s to d, consider a most loaded resource in d. Then, all the players who play any other resource in d form a $\mathrm{DirConSet}$ of size at most $n - \lceil n/m \rceil$.

For tightness, intuitively, consider an example, where the self-movement after the controlled players have moved is towards just one resource.

Therefore, if $s_i \neq d_i, \forall i \in N$, then $\mathrm{DirConNum} \geq n - \lceil n/m \rceil$. □

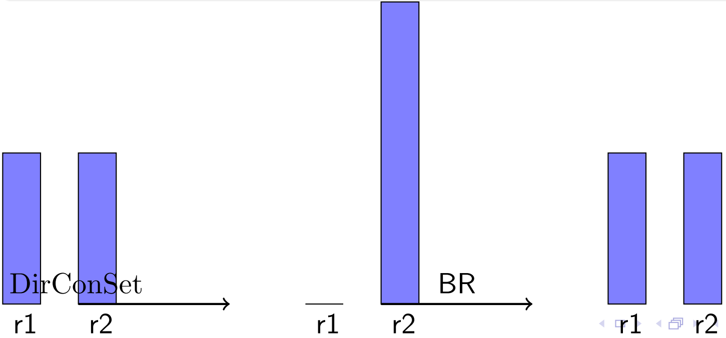
Congestion Games - Singleton

The formal proof requires some definitions.

First, denote $P_r(x) \triangleq \{i \in N : x_i = \{r\}\}$, and for any $A \subseteq R$, let $P_A(x) \triangleq \bigcup_{r \in A} P_r(x)$.

Definition (Intermediate profile, applies to any game)

Given a game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, profiles $s, d \in S$, and a direct control set $E \subseteq N$, denote the intermediate profile $(d_E, s_{N \setminus E})$ by $sd_G(E)$.

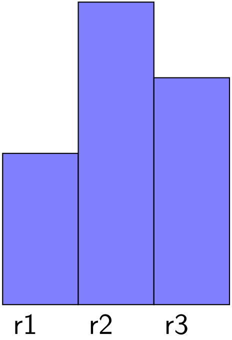


Congestion Games - Singleton - Defs

Definition (Attraction basin, applies to singleton congestion games)

Given a singleton congestion game $(N, R, (S_i)_{i \in N}, (c_j)_{j \in R})$ *and* $s \in S$, *define the attraction basin of* s *in* G *as* $AB_G(s) \stackrel{\Delta}{=} \bigcup_{i \in N} (\cup_{x_i \in Br_i(s_{-i})} x_i \setminus s_i)$.

Here, for example, if all the costs are $c(x) \stackrel{\Delta}{=} x$, then $AB = \{r1\}$.



Congestion Games - Singleton

Theorem

For singleton congestion games with n players and m resources, there always exists a $\mathrm{DirConSet}$ *of size at most* $n - \lceil n/m \rceil$*, and this is tight.*

A formal tight example.

Proof.

Consider a singleton congestion game G with $c_j(x) \triangleq m(x-1)+j$. Let s and d fulfil $s_i \neq d_i, \forall i \in N$, and $\lfloor n/m \rfloor \leq |P_r(d)| \leq \lceil n/m \rceil$, for any $r \in R$. In any profile, no two resources have equal costs $\Rightarrow |AB(sd(E))| \leq 1$, for any $E \subseteq N$. All the players in $N \setminus P_{AB(sd(E))}$ need to move, therefore they all have to belong to the $\mathrm{DirConSet} \Rightarrow |\mathrm{DirConSet}| \geq n - \lceil n/m \rceil$. □

Congestion Games - Singleton - Algorithm

Assumption (General position)

For any profile $s \in S, |AB(s)| \leq 1$.

Assuming $AB(sd(E)) = \{r\}$, any optimum $\mathrm{DirConSet}(G, s, d)$ is:

$$\{\text{move from } r\} \cup \{\text{move from } p \neq r \text{ to } q \neq r\} \tag{2}$$
$$\cup \{\text{stay at } p \neq r, \text{ while the BR is } \{r\}\} \tag{3}$$
$$\cup \{\text{all } P_r(d) \setminus P_r(s) \text{ OR all but one } P_r(d) \setminus P_r(s) \text{ OR } \emptyset\}. \tag{4}$$

This is a polynomial number of options.
It remains to prove that if more than one of $P_r(d) \setminus P_r(s)$ are missing, then r is the unique BR for everyone.
Suffices to prove $c_p(l_p(d)) > c_r(l_r(d) - 1), \forall p \neq r$.

Conclusions

1. Modelling improving equilibria through agents, internally
2. Always reaches the desired stable set

Future Work

- Order-Independence model, to keep the agents always motivated
- Gradual changes and learning
- Subsidising the control set? Cooperative approaches?

Thank You!